

# Wannabe modularity, the Shintani-Faddeev cocycle, and Stark units

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## The $q$ -Pochhammer symbol

The (infinite)  $q$ -Pochhammer symbol is

$$(w, q)_{\infty} = \prod_{k=0}^{\infty} (1 - wq^k).$$

Write  $w = e(z) = e^{2\pi iz}$  and  $q = e(\tau) = e^{2\pi i\tau}$ , and set

$$\varpi(z, \tau) = (w, q)_{\infty}.$$

### Question

How does  $\varpi(z, \tau)$  transform under:

- elliptic transformations  $z \mapsto z + m\tau + n$ ?
- modular transformations  $(z, \tau) \mapsto \left( \frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right)$ ?

## The Jacobi triple product and theta functions

The Jacobi triple product says that

$$\prod_{k=1}^{\infty} (1 - q^k)(1 - wq^{k-\frac{1}{2}})(1 - w^{-1}q^{k-\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} w^n q^{\frac{n^2}{2}}.$$

After a change of variables, the Jacobi triple product is equivalent to the formula

$$\varpi(z, \tau)\varpi(-z, \tau) = -ie\left(-\frac{\tau}{12}\right) \left(e\left(\frac{z}{2}\right) - e\left(-\frac{z}{2}\right)\right) \frac{\vartheta_1(z, \tau)}{\eta(\tau)}, \text{ where}$$

$\eta(\tau) = e\left(\frac{\tau}{24}\right) \prod_{k=1}^{\infty} (1 - e(k\tau))$  is the Dedekind eta function, and

$$\vartheta_1(z, \tau) = - \sum_{n=-\infty}^{\infty} e\left(\frac{1}{2} \left(n + \frac{1}{2}\right)^2 \tau + \left(n + \frac{1}{2}\right) \left(z + \frac{1}{2}\right)\right)$$

is a Jacobi theta function.

## The Jacobi triple product and theta functions

$$\varpi(z, \tau)\varpi(-z, \tau) = -ie\left(-\frac{\tau}{12}\right) \left(e\left(\frac{z}{2}\right) - e\left(-\frac{z}{2}\right)\right) f(z, \tau),$$

$$\text{where } f(z, \tau) = \frac{\vartheta_1(z, \tau)}{\eta(\tau)}.$$

- The function  $f(z, \tau)$  is a meromorphic Jacobi form of weight 0 with character—it transforms “nicely” under elliptic and modular transformations.
- $\varpi(z, \tau)$  is essentially “half” of a meromorphic Jacobi form—does it have interesting transformation properties on its own?

## Elliptic transformations of $\varpi(z, \tau)$

$$f(z, \tau) = \frac{\vartheta_1(z, \tau)}{\eta(\tau)}; \varpi(z, \tau) = (e(z), e(\tau))_\infty.$$

$$f(z+1, \tau) = -f(z, \tau) \qquad \varpi(z+1, \tau) = \varpi(z, \tau)$$

$$f(z+\tau, \tau) = -e\left(-\frac{\tau}{2} - z\right) f(z, \tau) \qquad \varpi(z+\tau, \tau) = (1 - e(z))^{-1} \varpi(z, \tau)$$

General elliptic transformation: For any  $m, n \in \mathbb{Z}$ ,

$$f(z + m\tau + n, \tau) = (-1)^{m+n} e\left(-\frac{m^2}{2}\tau - z\right) f(z, \tau)$$

$$\varpi(z + m\tau + n, \tau) = (e(z), e(\tau))_m^{-1} \varpi(z, \tau)$$

## Modular transformations of $\varpi(z, \tau)$

$SL_2(\mathbb{Z}) = \langle S, T \rangle$ , where  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

$S \cdot (z, \tau) = \left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$  and  $T \cdot (z, \tau) = (z, \tau + 1)$ .

Transformations under  $T$ :

$$f(z, \tau + 1) = e\left(\frac{1}{8}\right) f(z, \tau) \quad \varpi(z, \tau + 1) = \varpi(z, \tau)$$

Transformations under  $S$ :

$$f\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -ie\left(\frac{z^2}{2\tau}\right) f(z, \tau)$$

$$\varpi\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (???)\varpi(z, \tau)$$

## Modular transformations of $\varpi(z, \tau)$ , continued

### Theorem (Shintani, 1977 (rephrased))

The modular transformation law for  $q$ -Pochhammer symbol under the  $S$ -matrix is

$$\varpi\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = e\left(\frac{\tau - 3 + \tau^{-1}}{24} + \frac{(\tau - z)(1 - z)}{4\tau}\right) \left(1 - e\left(\frac{z}{\tau}\right)\right) \cdot \text{Sin}_2(z, \tau) \varpi(z, \tau),$$

where  $\text{Sin}_2(z, \tau)$  is the **double sine function**, a meromorphic function of  $z \in \mathbb{C}$  and  $\tau \in \mathbb{C} \setminus (-\infty, 0]$ .

- Shintani called this result a “Proposition”.
- The term “double sine function” is due to Kurokawa. (Shintani called the function  $F$ .)

## Double sine function

Shintani's definition of  $\text{Sin}_2(z, \tau)$  is in terms of functions previously defined by Barnes. The **double zeta function** is

$$\zeta_2(\mathbf{s}, z; \omega_1, \omega_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (z + \omega_1 m + \omega_2 n)^{-\mathbf{s}}.$$

The **double gamma function** is

$$\Gamma_2(z; \omega_1, \omega_2) = \rho_2(\omega_1, \omega_2) \exp\left(\frac{d}{ds} \zeta_2(\mathbf{s}, z; \omega_1, \omega_2) \Big|_{s=0}\right).$$

The **double sine function** is

$$\text{Sin}_2(z; \omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - z; \omega_1, \omega_2)}{\Gamma_2(z; \omega_1, \omega_2)}.$$

Because  $\text{Sin}_2(\alpha z; \alpha \omega_1, \alpha \omega_2) = \text{Sin}_2(z; \omega_1, \omega_2)$ , we lose no generality by defining  $\text{Sin}_2(z, \tau) = \text{Sin}_2(z; \tau, 1)$ .



## History—number theory side

- Shintani (1977) used his **double sine function** to give “Kronecker limit formulas” for first derivatives at zero of partial zeta functions attached to real quadratic number fields, as products of double sine values.
- Shintani also generalized to totally real fields.
- Kurowawa and Koyama studied these functions further and found other number-theoretic applications.
- Zagier, Hayes, Sczech, Tangedal, Yamamoto, and others have refined or reinterpreted Shintani’s formulas.

## History—physics side

- Meanwhile, Faddeev (1994) independently discovered the double sine function and its relation to the  $q$ -Pochhammer symbol. He called it the **noncompact quantum dilogarithm** and used it in conformal field theory.
- The physics literature includes many fascinating and highly nontrivial integral formulas involving the noncompact quantum dilogarithm.
- Recently, Sarkissian and Spiridonov (2020) considered a **general modular quantum dilogarithm**.

## General modular transformations of $\varpi(z, \tau)$

### Theorem (K 2022+, Sarkissian-Spiridonov 2020)

For each  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , there is a meromorphic function  $\sigma_\gamma(z, \tau)$  on  $\mathbb{C} \times (\mathbb{C} \setminus \{\tau \in \mathbb{R} : c\tau + d \leq 0\})$  with the property that

$$\varpi\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \sigma_\gamma(z, \tau)\varpi(z, \tau).$$

### Definition (K 2022+)

We call the map  $\gamma \mapsto \sigma_\gamma(z, \tau)$  the **Shintani-Faddeev Jacobi cocycle**.

## So...

Questions:

- In what sense is  $\gamma \mapsto \sigma_\gamma(\mathbf{z}, \tau)$  a cocycle?
- In what sense is  $\varpi(\mathbf{z}, \tau)$  modular or Jacobi-form-like?

I'm still trying to find the best answers to these questions—but I will present my current working formalism.

# Generalized group cohomology

Suppose  $\Gamma$  is a group and  $A$  is a  $\mathbb{Z}[\Gamma]$ -module. The first cohomology group is

$$H^1(\Gamma, A) = \frac{\{a : \Gamma \rightarrow A \mid a_{\gamma_1\gamma_2} = a_{\gamma_1}^{\gamma_2} a_{\gamma_2}\}}{\{a : \Gamma \rightarrow A \mid a_\gamma = c^\gamma c^{-1} \text{ for some } c \in A\}}.$$

## Definition (K 2022+)

If we also have a function  $B : \Gamma \rightarrow \{\text{subgroups of } A\}$ , we may define a **generalized first cohomology group**

$$H_B^1(\Gamma, A) = \frac{\{a : \Gamma \rightarrow A \mid a_\gamma \in B_\gamma, a_{\gamma_1\gamma_2} = a_{\gamma_1}^{\gamma_2} a_{\gamma_2}\}}{\{a : \Gamma \rightarrow A \mid a_\gamma = c^\gamma c^{-1} \text{ for some } c \in \bigcap_{\gamma \in \Gamma} B_\gamma\}}.$$

## Weight cocycles

Now let  $\Gamma$  be a discrete subgroup of  $SL_2(\mathbb{R})$ .

### Definition (K 2022+)

A **system of domains** is a collection of connected open sets  $\mathbb{H} \subseteq D_\gamma \subseteq \mathbb{C} \cup \{\infty\}$  indexed by  $\gamma \in \Gamma$ . A **weight cocycle** (or “generalized factor of automorphy”) is a collection of nonzero meromorphic functions  $w_\gamma : D_\gamma \rightarrow \mathbb{C}$  such that

$$w_{\gamma_1\gamma_2}(\tau) = w_{\gamma_1}(\gamma_2 \cdot \tau) w_{\gamma_2}(\tau).$$

In other words,  $w_\gamma$  defines a cohomology class

$$[w_\gamma] \in H_B^1(\Gamma, \mathcal{M}_{\mathbb{H}}^\times)$$

where  $\mathcal{M}_D$  denotes the ring of meromorphic functions on  $D$ , and  $\mathcal{B}_\gamma = \mathcal{M}_{D_\gamma}^\times$ .

## Wannabe modular forms



# Wannabe modular forms

## Definition (K 2022+)

Let  $w_\gamma(\tau)$  be a weight cocycle for some system of domains  $D_\gamma$ . A meromorphic complex-valued function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is **wannabe modular of weight  $w_\gamma(\tau)$**  if

$$f(\gamma \cdot \tau) = w_\gamma(\tau)f(\tau)$$

for all  $\tau \in \mathbb{H}$ .

**First example:** For every  $k \in \mathbb{Z}$ , the function  $j_\gamma(\tau)^k = (c\tau + d)^k$  is a weight cocycle for the constant system of domains  $D_\gamma = \mathbb{C} \cup \{\infty\}$ , and any modular form of weight  $k$  is a wannabe modular form of weight  $j_\gamma(\tau)^k$ .



# The Shintani-Faddeev modular cocycle

**Second example:** For  $\mathbf{r} = (r_1, r_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$ , let

$$\varpi_{\mathbf{r}}(\tau) = \varpi(r_1\tau + r_2, \tau) = (e(r_1\tau + r_2), e(\tau))_{\infty}.$$

## Theorem (K 2022+)

The function  $\varpi_{\mathbf{r}}(\tau)$  is a wannabe modular form for a congruence subgroup of level  $N$ . The associated weight cocycle  $\gamma \mapsto \psi_{\gamma}^{\mathbf{r}}(\tau)$ , called the **Shintani-Faddeev modular cocycle (with characteristics  $\mathbf{r}$ )**, is valued in meromorphic functions  $\psi_{\gamma}^{\mathbf{r}}(\tau)$  on  $\tilde{D}_{\gamma}$ , where

$$\tilde{D}_{\gamma} = \begin{cases} \mathbb{C} \setminus (-\infty, -d/c] & \text{if } c > 0, \\ \mathbb{C} & \text{if } c = 0, \\ \mathbb{C} \setminus [-d/c, \infty) & \text{if } c < 0. \end{cases}$$

## Wannabe Jacobi forms

We may define wannabe Jacobi forms in a similar manner (details omitted).

### Theorem (rephrasing of transformation laws)

The function  $\varpi(z, \tau)$  is a wannabe Jacobi form with weight cocycle  $\gamma \mapsto \sigma_\gamma(z, \tau)$  and system of domains

$$D_\gamma = \begin{cases} \mathbb{C} \setminus (-\infty, -d/c] & \text{if } c > 0, \\ \mathbb{C} & \text{if } c = 0 \text{ and } d > 0, \\ \mathbb{H} & \text{if } c = 0 \text{ and } d < 0, \\ \mathbb{C} \setminus [-d/c, \infty) & \text{if } c < 0. \end{cases}$$

# Values of the Shintani-Faddeev cocycle at $\tau = \frac{m}{n}$ and quantum modularity

## Proposition (K 2022+)

Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , and let  $m, n \in \mathbb{Z}$ ,  $n > 0$  such that  $\mathrm{gcd}(m, n) = 1$  and  $c\frac{m}{n} + d > 0$ . For  $\mathrm{Im}(z) > 0$ ,

$$\sigma_\gamma\left(z, \frac{m}{n}\right) = \frac{D_{e\left(\frac{am+bn}{cm+dn}\right)}\left(e\left(\frac{nz}{cm+dn}\right)\right)^{-1/(cm+dn)}}{D_{e\left(\frac{m}{n}\right)}(e(z))^{-1/n}},$$

where  $D_\zeta(w) = \prod_{k=1}^{n-1} (1 - \zeta^k w)^k$  for  $\zeta = e\left(\frac{m}{n}\right)$ .

- $D_\zeta(w)$  is called the **cyclic quantum dilogarithm** and plays a role in Garoufalidis and Zagier's work on Nahm's conjecture.
- The proposition above **almost** implies that  $f\left(z, \frac{m}{n}\right) = -\frac{1}{n} \log D_{e\left(\frac{m}{n}\right)}(e(z))$  defines a quantum Jacobi form of weight 0 in the sense of Bringmann and Folsom.

## Hilbert's 12 problem and the Stark conjectures

- Kronecker-Weber theorem says that the abelian Galois extensions of  $\mathbb{Q}$  are generated by the values of  $e(z) = e^{2\pi iz}$  at rational values of  $z$ .
- Hilbert's 12th problem asks for an extension of Kronecker-Weber where  $\mathbb{Q}$  is replaced by a general number field  $K$ .
- Given any  $K$ , Hilbert wanted analytic functions that play the role of  $e(z)$ .
- Harold Stark conjectured in a series of papers (1971–1980) that  $\exp(cZ(1))$ , for certain linear combinations  $Z(s)$  of Hecke  $L$ -functions of  $K$ , generate abelian extensions of  $K$ .

## $L$ -functions at $s = 1$ : rational example

The following formula can be proved using calculus. Try it!

### Example

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + \dots = \frac{1}{\sqrt{2}} \log(1 + \sqrt{2})$$

The left-hand side is the value  $L(1, \chi)$ , where  $\chi(n) = \left(\frac{2}{n}\right)$  is the Dirichlet character associated to the field extension  $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ . The right-hand side involves  $\varepsilon = 1 + \sqrt{2}$ , the fundamental unit of  $\mathbb{Q}(\sqrt{2})$ .

## $L$ -functions at $s = 1$ : imaginary quadratic example

The following formula is proved using the theory of complex multiplication for elliptic curves.

### Example

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \sum \frac{e(m/5) - e(2m/5)}{m^2 + mn + n^2} = \frac{2\pi}{\sqrt{3}} \log(\varepsilon^{1/5})$$

where  $\varepsilon = 29 + 12\sqrt{5} + 2\sqrt{6(65 + 29\sqrt{5})}$ .

The left-hand side is a linear combination of Hecke  $L$ -values at  $s = 1$  for  $\mathbb{Q}(\sqrt{-3})$ . The right-hand side involves an algebraic unit  $\varepsilon$  in the ray class field modulo (5) for  $\mathbb{Q}(\sqrt{-3})$ .

This example is related to the 5-torsion points of the CM elliptic curve  $y^2 = x^3 + 1$ . This elliptic curve has complex multiplication by  $\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right]$ .

## $L$ -functions at $s = 1$ : real quadratic example

The following formula is an open conjecture!

### Example

$$\sum_{m=1}^{\infty} \sum_{\substack{n \in \mathbb{Z} \\ -\frac{5}{3}m \leq n < \frac{5}{3}m}} \frac{e(4m/5) - e(m/5)}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log(\varepsilon),$$

where  $\varepsilon \approx 3.890861714$  is a root of the polynomial equation

$$\begin{aligned} x^8 - (8 + 5\sqrt{3})x^7 + (53 + 30\sqrt{3})x^6 - (156 + 90\sqrt{3})x^5 \\ + (225 + 130\sqrt{3})x^4 - (156 + 90\sqrt{3})x^3 + (53 + 30\sqrt{3})x^2 \\ - (8 + 5\sqrt{3})x + 1 = 0. \end{aligned}$$

The number  $\varepsilon$  is an algebraic unit in the ray class field of  $\mathbb{Q}(\sqrt{3})$  modulo  $5\infty_2$ . This conjecture is part of the Stark conjectures.

## Ray class groups and ray class fields

Let  $K$  be a number field and  $\mathcal{O}_K$  its ring of integers. Let  $\mathfrak{m}$  be a ideal in  $\mathcal{O}_K$ , and let  $\Sigma$  be a subset of the real embeddings of  $K$ .

### Definition (Ray class group modulo $\mathfrak{m}, \Sigma$ )

$$\text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K) = \frac{\{\text{fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{m}\}}{\{a\mathcal{O}_K \text{ s.t. } a \equiv 1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}}$$

Class field theory associates to  $\text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$  a **ray class field**  $H_{\mathfrak{m}, \Sigma}$ , an abelian extension of  $K$  with Galois group  $\text{Gal}(H_{\mathfrak{m}, \Sigma}/K) = \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$ . Varying  $\mathfrak{m}$  and  $\Sigma$ , the ray class fields are cofinal among all abelian extensions of  $K$ .



## Zeta functions associated to ray classes

### Definition

For  $\mathcal{A} \in \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$ , the associated zeta function is

$$\zeta(\mathfrak{s}, \mathcal{A}) = \sum_{\substack{\mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{a} \in \mathcal{A}}} \text{Nm}(\mathfrak{a})^{-\mathfrak{s}}.$$

Let  $\mathcal{R} \in \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$  be the ideal class

$$\mathcal{R} = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{m}} \text{ and } \rho(a) > 0 \text{ for } \rho \in \Sigma\}.$$

### Definition

For  $\mathcal{A} \in \text{Cl}_{\mathfrak{m}, \Sigma}(\mathcal{O}_K)$ , the associated differenced zeta function is

$$Z_{\mathcal{A}}(\mathfrak{s}) = \zeta(\mathfrak{s}, \mathcal{A}) - \zeta(\mathfrak{s}, \mathcal{R}\mathcal{A}).$$

## Rank 1 abelian Stark conjecture over a real quadratic field

## Conjecture (Stark, 1976, special case)

Setup:

- Let  $K$  be a real quadratic number field.
- Consider  $0 \neq \mathfrak{m} \leq \mathcal{O}_K$  with the property that, if  $\varepsilon \in \mathcal{O}_K^\times$  and  $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$ , then one of  $\varepsilon$  or  $-\varepsilon$  is totally positive.
- Let  $\mathcal{A}$  be a ray ideal class in  $\text{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_K)$ .
- Let  $H_{\mathfrak{m}\infty_j}$  be the ray class field of  $K$  modulo  $\mathfrak{m}\infty_j$ .
- Let  $\rho_j$  be the real embedding of  $H_{\mathfrak{m}\infty_1\infty_2}$  associated to  $\infty_j$ .

Then,

- (1)  $Z'_{\mathcal{A}}(0) = \log(\rho_1(\varepsilon_{\mathcal{A}}))$  for a unit  $\varepsilon_{\mathcal{A}} \in H_{\mathfrak{m}\infty_2}$ .
- (2) The units  $\varepsilon_{\mathcal{A}}$  are compatible with the isomorphism  $\text{Art} : \text{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_K) \rightarrow \text{Gal}(H_{\mathfrak{m}\infty_2}/K)$ . Specifically,  $\varepsilon_{\mathcal{A}} = \varepsilon_{\text{id}}^{\text{Art}(\mathcal{A})}$ .

## Hilbert's 12th problem and cocycles

- Hilbert's 12th problem was solved for imaginary quadratic fields using complex multiplication values of modular functions (0-cocycles for congruence subgroups of  $SL_2(\mathbb{Z})$ ).
- **Hope:** Use “real multiplication values” of 1-cocycles to solve Hilbert's 12th problem for real quadratic fields.
- In fact, it's no longer just a hope: Darmon, Pozzi, and Vonk (2021) prove algebraicity results for real multiplication values of their  $p$ -adic Dedekind-Rademacher cocycle using the groundbreaking work of Dasgupta and Kakde (2021) on the Gross-Stark and Brumer-Stark conjectures.
- But what about the complex (not  $p$ -adic) setting?

## Real multiplication values of a weight cocycle

- Let  $w_\gamma(\tau)$  be a weight cocycle.
- Consider a real quadratic number  $\beta$ .
- Suppose  $\gamma \in \Gamma$  is the “positive” generator for the stabilizer of  $\beta$  in  $\Gamma$ .
- Then, the values  $w_\gamma(\beta)$  is the **real multiplication value** of  $w$  at  $\beta$ .

Real multiplication values of  $\psi_\gamma^{\mathbf{p}}(\tau)$ 

## Theorem (K 2022+)

Let  $K$  be a real quadratic field,  $N \in \mathbb{N}$ . Let  $\mathcal{A} \in \text{Cl}(\mathcal{O}_K)$ , choose some  $\mathfrak{b} \in \mathcal{A}^{-1}$  coprime to  $N\mathcal{O}_K$ , and write  $\mathfrak{b} = \mathbb{Z} + \beta\mathbb{Z}$  for some  $\beta \in K$ . Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be the positive generator for  $\text{stab}(\beta)$  in  $\Gamma(N)$ . Let  $\beta_0 \in \mathcal{O}_K$  such that  $\beta \equiv \beta_0 \pmod{N}$ . For  $\mathbf{p} = (p_1, p_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ , let

$$\mathcal{A}_{\mathbf{p}} = \{\alpha\mathcal{O}_K : \alpha \equiv p_1\beta + p_2 \pmod{N} \text{ and } \rho_2(\alpha) > 0\} \in \tilde{\text{Cl}}_{N\infty_2}(\mathcal{O}_K).$$

For an easily computable integer  $n (= 1 \text{ if } \beta_0\mathcal{O}_K + N\mathcal{O}_K = \mathcal{O}_K)$ ,

$$\begin{aligned} & n \exp(Z'_{N\infty_2}(0, \mathfrak{b}^{-1}\mathcal{A}_{\mathbf{p}})) \\ &= \left( \zeta_8^{-1} \epsilon(\gamma) e\left(\frac{N+1}{4} (bp_1^2 - (a-d)p_1p_2 - cp_2^2)\right) \psi_\gamma^{N^{-1}\mathbf{p}}(\beta) \right)^2. \end{aligned}$$

## Proof outline

- Use Jacobi triple product and modular properties of  $\vartheta_1(z, \tau)$  and  $\eta(\tau)$  to establish a relation between  $\psi_\gamma^r(\tau)$  and  $\psi_\gamma^{-r}(\tau)$ .
- Write Tangedal's version of Shintani's formula (involving the Hirzebruch-Jung continued fraction of  $\beta$ ) in terms of the Shintani-Faddeev cocycle, and use the cocycle condition to "telescope" the product.
- One is left with a complicated-looking root of unity factor...
- ...that may be simplified greatly using the combinatorics of continued fraction expansions and the relation between  $\psi_\gamma^r(\beta)$  and  $\psi_\gamma^{-r}(\beta)$  previously established.
- The maximal order  $\mathcal{O}_K$  may also be replaced by an arbitrary order  $\mathcal{O}$ .

## Future directions

- Big picture: Want to prove an algebraicity result for  $\psi_\gamma^r(\beta)$ .
- Work out (conjecturally, at least) the action of the Artin map on the sign under the square root.
- Understand cohomology group containing  $\gamma \mapsto \psi_\gamma^r(\tau)$ , that is,  $H_{\tilde{\mathcal{B}}_\gamma}^1(\Gamma(N), \mathcal{M}_{\mathbb{H}}^\times)$  with  $\tilde{\mathcal{B}}_\gamma = \mathcal{M}_{\tilde{D}_\gamma}^\times$  (or a smaller group, if this turns out to be too big).
- Connect complex and  $p$ -adic cocycles.
- Try to do something with formulas and perspectives originating in conformal field theory literature. Connect CFT to CFT?

Thank you!

Thank you for listening! Any questions?