Gauss composition, polyharmonic Maass forms, and Hecke L-series

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Origins of the subject

Let $Q(x, y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{Z}$.

Question

Which primes p are of the form $p = Q(x, y)$ for some $x, y \in \mathbb{Z}$?

Fermat: $p = x^2 + y^2$ if and only if $p = 2$ or $p \equiv 1 \pmod 4$.

Other $Q(x, y)$? Gauss studied this in Disquisitiones Arithmeticae.

Spaces of binary quadratic forms

Let $D \equiv 0, 1 \pmod 4$ be nonsquare.

$$\mathcal{Q}^{+}_{\text{prim}}(D) := \left\{ \begin{array}{c} Q(x,y) = ax^2 + bxy + cy^2 : a, b, c \in \mathbb{Z}, \\ b^2 - 4ac = D, \ \gcd(a,b,c) = 1, \\ \text{and } Q \text{ is not negative-definite} \end{array} \right\}.$$

Important fact: The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}^{+}_{\text{prim}}(D)$ by

$$Q^{\gamma}(x,y) = Q(rx+sy, tx+uy)$$

for $\gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{SL}(\mathbb{Z})$.

An equivalence class in $\mathcal{Q}_{\text{prim}}(D)^{+} / \mathrm{SL}_2(\mathbb{Z})$ is denoted by $[Q]$.

Representing primes

If p is an odd prime and $Q \in \mathcal{Q}^+_{\mathrm{prim}}(D)$, then

$$Q(m, n) = p \implies D \equiv \square \pmod{p}.$$

Theorem (Gauss)

Every odd prime p such that $\left(\frac{D}{p}\right) = 1$ is represented by exactly one class in $\mathcal{Q}^+_{\mathrm{prim}}(D)/\operatorname{GL}_2(\mathbb{Z})$.

Example ($D = -47$)

$[x^2+xy+12y^2]$	$\left\{ \begin{array}{l} [2x^2+xy+6y^2] \\ [2x^2-xy+6y^2] \end{array} \right\}$	$\left\{ \begin{array}{l} [3x^2+xy+4y^2] \\ [3x^2-xy+4y^2] \end{array} \right\}$
47, 83, 191, 197, ...	2, 7, 53, 59, 61, 89, 97, 131, 157, 173, ...	3, 17, 37, 71, 79, 101, 103, 149, ...

Gauss composition

Let $Q_1, Q_2 \in \mathcal{Q}^+_{\mathrm{prim}}(D)$. There exists some (non-unique) $Q_3 \in \mathcal{Q}^+_{\mathrm{prim}}(D)$ such that

$$Q_3(X, Y) = Q_1(x_1, y_1)Q_2(x_2, y_2)$$

where

$$X = Ax_1x_2 + Bx_1y_2 + Cy_1x_2 + Dy_1y_2,$$
$$Y = Ex_1x_2 + Fx_1y_2 + Gy_1x_2 + Hy_1y_2.$$

Theorem (Gauss)

$[Q_3]$ is uniquely determined by $[Q_1]$ and $[Q_2]$, and setting $[Q_1] \cdot [Q_2] = [Q_3]$ defines an abelian group law on $\mathcal{Q}^+_{\mathrm{prim}}(D)/\mathrm{SL}_2(\mathbb{Z})$.

Orders of quadratic fields

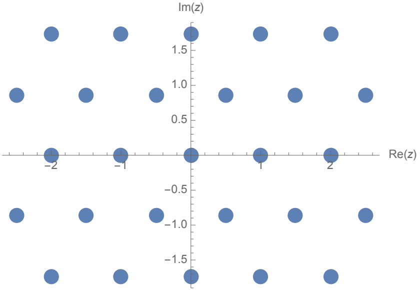
Modern interpretation of Gauss composition: ring class groups.

Definition

For $D = f^2 D_0$, $\mathcal{O}_D$ is the order of discriminant D:

$$\mathcal{O}_D := \mathbb{Z} + \frac{D+\sqrt{D}}{2}\mathbb{Z}.$$

Example:
$D = -3$



Multiplicative structure

Question

Does $\mathcal{O}_D$ have unique factorization into primes? Not generally.

Examples

$$2 \times 3 = (1 - \sqrt{-5})(1 + \sqrt{-5}) \text{ in } \mathcal{O}_{-20}$$

$$3^2 \times 7 = (4 + \sqrt{79})(4 - \sqrt{79}) \text{ in } \mathcal{O}_{316}$$

$$5^2 = (4 + 3i)(4 - 3i) \text{ in } \mathcal{O}_{-36}$$

Invertible ideals

Instead of numbers, look at invertible ideals.

A fractional ideal $\mathfrak{a}$ of $\mathcal{O}_D$ is invertible if there is another fractional ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}_D$.

Invertible ideals of $\mathcal{O}_D$ always enjoy unique factorization into prime ideals.

A fractional ideal is principal if it is of the form $\mathfrak{a} = \alpha\mathcal{O}_D$.

Nonprincipal ideals obstruct unique factorization.

Example: $D = -36$

$$5^2 = (4+3i)(4-3i) \text{ in } \mathcal{O}_{-36}$$

These irreducible numbers factor as ideals:

$$5\mathcal{O}_{-36} = \mathfrak{p}\mathfrak{p}' \quad (4+3i)\mathcal{O}_{-36} = \mathfrak{p}^2 \quad (4-3i)\mathcal{O}_{-36} = (\mathfrak{p}')^2;$$

$$\mathfrak{p} = (6-3i)\mathcal{O}_{-36} + (4+3i)\mathcal{O}_{-36};$$
$$\mathfrak{p}' = (6+3i)\mathcal{O}_{-36} + (4-3i)\mathcal{O}_{-36}.$$

To understand arithmetic of $\mathcal{O}_D$, study its nonprincipal ideals.

Avoid noninvertible ideals, such as $3\mathcal{O}_{-36} + 3i\mathcal{O}_{-36}$.

Class groups and class numbers

Definition

The ring class group of $\mathcal{O}_D$ is

$$\mathrm{Cl}(\mathcal{O}_D) := \frac{\{\text{invertible fractional ideals of } \mathcal{O}_D\}}{\{\text{principal fractional ideals } \alpha\mathcal{O}_D\}}.$$

- $|\mathrm{Cl}(\mathcal{O}_D)| = 1$ if and only if $\mathcal{O}_D$ has unique factorization of numbers coprime to the conductor f.

Narrow ring class group

Definition

The narrow ring class group of $\mathcal{O}_D$ is

$$\mathrm{Cl}^+(\mathcal{O}_D) = \frac{\{\text{invertible fractional ideals of } \mathcal{O}_D\}}{\{\alpha\mathcal{O}_D \text{ with } \mathrm{Nm}(\alpha) > 0\}}.$$

Theorem (Gauss, Dirichlet, Dedekind)

$$\mathrm{Cl}^+(\mathcal{O}_D) \cong \mathcal{Q}^+_{\mathrm{prim}}(D)/\,\mathrm{SL}_2(\mathbb{Z}).$$

Class field theory

Another interpretation of $\mathcal{Q}^+_{\text{prim}}(D)/\operatorname{SL}_2(\mathbb{Z})$: Galois group.

Question

What are the abelian extensions of K?

- If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_D$, then $\mathfrak{p}$ ramifies in an extension of K if $\mathfrak{p}$ is divisible by the square of a prime ideal in the extension.

Theorem (part of Artin Reciprocity)

$$\text{Art} : \text{Cl}^+(\mathcal{O}_D) \cong \text{Gal}(H_D^+/K),$$

where H_D^+ is the maximal abelian extension of $\mathbb{Q}(\sqrt{D})$ that is unramified at every prime ideal of $\mathcal{O}_D$.

Representing primes

Let ϕ be the isomorphism from $\mathcal{Q}^+_{\mathrm{prim}}(D)/\operatorname{SL}_2(\mathbb{Z})$ to $\mathrm{Cl}^+(\mathcal{O}_D)$.

Corollary of Artin reciprocity and Gauss composition

Let p be a rational prime with $\gcd(p, D) = 1$, $Q \in \mathcal{Q}^+_{\mathrm{prim}}(D)$. TFAE:

(1) $Q(m, n) = p$ for some $m, n \in \mathbb{Z}$.

(2) $\phi(Q) = [\mathfrak{p}]$ in $\mathrm{Cl}^+(\mathcal{O}_D)$, where $(p) = \mathfrak{p}\mathfrak{p}'$ in $\mathcal{O}_D$.

(3) $\mathrm{Art}(\phi(Q)) = \mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{Gal}\left(H_D^+/K\right)$, where $(p) = \mathfrak{p}\mathfrak{p}'$ in $\mathcal{O}_D$.

Ray class groups

- Ring class fields do not generate all abelian extensions of a number field.
- To describe the Galois groups of all finite abelian extensions, we need ray class groups.

Definition (K and Lagarias for nonmaximal orders)

Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_D$ and $S \subseteq \{\text{real embeddings } K \hookrightarrow \mathbb{R}\}$.

$$\mathrm{Cl}_{\mathfrak{m},S}(\mathcal{O}_D) = \frac{\{\text{invertible fractional ideals of } \mathcal{O}_D \text{ coprime to } \mathfrak{m}\}}{\{\alpha\mathcal{O}_D \text{ with } \alpha \equiv 1 \pmod{\mathfrak{m}} \text{ and } \rho(\alpha) > 0 \text{ for } \rho \in S\}}.$$

Theorem (Artin Reciprocity + K and Lagarias)

There is an abelian extension $H_D^{\mathfrak{m},S}$ of K (uniquely specified by certain conditions on splitting of primes) with an isomorphism

$$\mathrm{Art} : \mathrm{Cl}_{\mathfrak{m},S}(\mathcal{O}_D) \cong \mathrm{Gal}(H_D^{\mathfrak{m},S}/K).$$

Narrow ray class groups

Let ∞_1, ∞_2 be the real embeddings of $K = \mathbb{Q}(\sqrt{D})$.

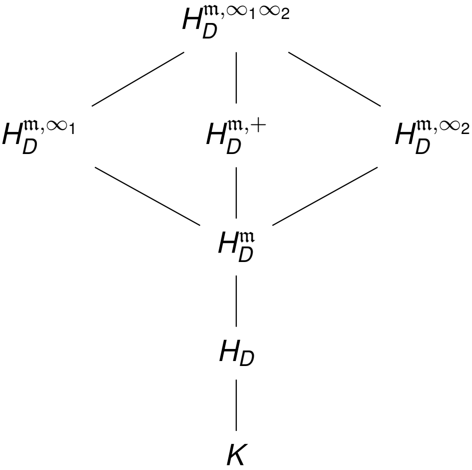
$$
\begin{array}{ccccc}
 & & \mathrm{Cl}_{\mathfrak{m}\infty_1\infty_2}(\mathcal{O}_D) & & \\
 & \swarrow & \downarrow & \searrow & \\
\mathrm{Cl}_{\mathfrak{m}\infty_1}(\mathcal{O}_D) & & \mathrm{Cl}^+_{\mathfrak{m}}(\mathcal{O}_D) & & \mathrm{Cl}_{\mathfrak{m}\infty_2}(\mathcal{O}_D) \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathrm{Cl}_{\mathfrak{m}}(\mathcal{O}_D) & &
\end{array}
$$

Definition

The narrow ray class group of $\mathcal{O}_D$ modulo $(\mathfrak{m}, S)$ is

$$\mathrm{Cl}^+_{\mathfrak{m}}(\mathcal{O}_D) = \frac{\{\text{invertible fractional ideals of } \mathcal{O} \text{ coprime to } \mathfrak{m}\}}{\{\alpha\mathcal{O}_D \text{ with } \alpha \equiv 1 \pmod{\mathfrak{m}} \text{ and } \mathrm{Nm}(\alpha) > 0\}}.$$

Field theoretic interpretation



Refined Gauss composition

$$\mathcal{Q}_{\mathrm{prim}}^{N,+}(D) := \left\{ \begin{array}{c} Q(x,y) = ax^2 + bxy + cy^2 : a,b,c \in \mathbb{Z}, \\ b^2 - 4ac = D,\ \gcd(a,N) = \gcd(a,b,c) = 1, \\ \text{and } Q \text{ is not negative-definite} \end{array} \right\}.$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} r & s \\ t & u \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Theorem 1 (Beckwith and K, 2021+)

There is a bijection $\phi : \mathcal{Q}_{\mathrm{prim}}^{N,+}(D)/\Gamma_1(N) \cong \mathrm{Cl}_{(N)}^{+}(\mathcal{O}_D)$. $\mathcal{Q}_{\mathrm{prim}}^{N,+}(D)/\Gamma_1(N)$ has an abelian group structure.

Known for $\mathcal{O}_{D_0}$ with fundamental discriminant $D_0 < 0$ by Eum, Koo, and Shin in 2017.

Mapping forms to ideals

$$Q(x,y) = ax^2 + bxy + cy^2 = a(x - \tau y)(x - \tau' y)$$

with $\tau = \frac{-b+\sqrt{D}}{2a}$. Define the twisting factor:

$$R_Q = \begin{cases} \{\alpha \mathcal{O}_D : \alpha \equiv 1 \pmod{N}, \mathrm{Nm}(\alpha) < 0\} & \text{if } D > 0 \text{ and } a < 0, \\ \mathrm{id}_N & \text{if } D < 0 \text{ or } a > 0, \end{cases}$$

Define $\phi(Q) = R_Q \left[a(\mathbb{Z} + \tau\mathbb{Z})\right]$.

One must then check that...

- $a(\mathbb{Z} + \tau\mathbb{Z})$ is coprime to N
- $a(\mathbb{Z} + \tau\mathbb{Z})$ is invertible,
- $\phi(Q^\gamma) \sim \phi(Q)$ in $\mathrm{Cl}^+_{(N)}(\mathcal{O}_D)$ for $\gamma \in \Gamma_1(N)$,
- $[Q] \mapsto [\phi(Q)]$ is injective and surjective.

Representation of primes

Theorem 2 (Beckwith and K, 2021+)

Let p be a rational prime, and suppose $\gcd(p, ND) = 1$. Fix a binary quadratic form $Q \in \mathcal{Q}_{\mathrm{prim}}^{N,+}(D)$. The following are equivalent:

(1) $Q(m, n) = p$ for some $(m, n) \equiv (1, 0) \pmod{N}$.

(2) $\phi(Q) = [\mathfrak{p}]$ in $\mathrm{Cl}_{(N)}^{+}(D)$, where $(p) = \mathfrak{p}\mathfrak{p}'$ for distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ in $\mathcal{O}_D$.

(3) $\mathrm{Art}(\phi(Q)) = \mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{Gal}(H_{(N)}^{\mathcal{O}_D,+}/K)$, where $(p) = \mathfrak{p}\mathfrak{p}'$ for distinct prime ideals $\mathfrak{p}$, $\mathfrak{p}'$ in $\mathcal{O}_D$.

Example

$D = -7$, $\mathcal{O}_D = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$, $N = 3$.

$\mathrm{Cl}_3^+(\mathcal{O}_{-7}) \cong \mathbb{Z}/4\mathbb{Z}$.

$[Q]$	$p = Q(m,n)$: $(m,n) \equiv (1,0) \pmod 3$
$[x^2 + xy + 2y^2]$	67, 79, 127, 163, 277, 373, 421, 463 ...
$[2x^2 + 3xy + 2y^2]$ $[2x^2 - 3xy + 2y^2]$	2, 11, 23, 29, 53, 71, 107, 113, 137, 149, 179, 191, 197, 233, 239, 263, 281, 317, 347, 359, 389, 401, 431, 443, 449, 491 ...
$[4x^2 + 5xy + 2y^2]$	7, 37, 43, 109, 151, 193, 211, 331, 337, 379, 457, 487, 499 ...

Example

$D = 21$, $\mathcal{O}_D = \mathbb{Z}[\frac{1+\sqrt{21}}{2}]$, $N = 6$.

$\mathrm{Cl}_6^+(\mathcal{O}_{21}) \cong \mathbb{Z}/6\mathbb{Z}$.

$[Q]$	$p = Q(m,n)$: $(m,n) \equiv (1,0) \pmod 6$
$[x^2 + 5xy + y^2]$	7, 67, 211, 421, 457, 487 ...
$[-(x^2 + 5xy + y^2)]$	89, 101, 131, 173, 227, 257, 467, 563, 587, ...
$[-5x^2 + xy + y^2]$ $[-5x^2 - xy + y^2]$	37, 43, 79, 109, 127, 151, 163, 193, 277, 331, 337, 373, 379, 463, 499, 541, 547, 571 ...
$[-(-5x^2 + xy + y^2)]$ $[-(-5x^2 - xy + y^2)]$	5, 17, 41, 47, 59, 83, 167, 251, 269, 293, 311, 353, 383, 419, 461,479, 503, 509, 521, 593 ...

Example

$D = 12$, $\mathcal{O}_D = \mathbb{Z}[\sqrt{3}]$, $N = 5$.
$\mathrm{Cl}_5^+\left(\mathbb{Z}[\sqrt{3}]\right) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

$[Q]$	$p = Q(m,n)$: $(m,n) \equiv (1,0) \pmod 5$
$[x^2 - 3y^2]$	61,181, 241 ...
$[-x^2 + 3y^2]$	59,179,239, 359 ...
$[3x^2 - y^2]$	3,23,83,263,383 ...
$[-3x^2 + y^2]$	37, 97, 157, 227, 397, ...
$[11x^2 - 34xy + 26y^2]$	11, 71, 131, 191, 251, 311, ...
$[-11x^2 + 34xy - 26y^2]$	109, 229, 349, ...
$[2x^2 - 2xy - y^2]$	2,47,107, 167, 227, 347 ...
$[-2x^2 + 2xy + y^2]$	13,73,193, 313, 373 ...

Segue

As an application of our results on refined Gauss composition (and other tools), we prove a formula for the leading coefficients of Hecke L-series for real quadratic fields as a “twisted trace" of biharmonic Maass forms.

Our motivation comes from explicit class field theory.

Class field theory

Let $K/\mathbb{Q}$ be a number field.

Question

What are the finite abelian extensions of K?

Class field theory

The finite abelian extensions of K correspond to quotients of the ray class groups for K.

Question

Is there an explicit description of these extensions?

Explicit class field theory

Kronecker-Weber Theorem

Every finite abelian extension of $\mathbb{Q}$ is contained in $\mathbb{Q}(e^{2\pi i/n})$ for some n.

Hilbert's Twelfth Problem

Find an analogue of the Kronecker-Weber theorem for number fields other than $\mathbb{Q}$.

In other words, find an explicit description of the finite abelian extensions of K, where K is a number field.

Kronecker's Jugendtraum

Let $K = \mathbb{Q}(\sqrt{D})$, where $D < 0$.

$\mathcal{O}_K = \mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$.
Let $j(z) : \mathbb{H} \to \mathbb{C}$ be the modular j-function,

$$j(\tau) = e^{-2\pi i\tau} + 744 + 196884e^{2\pi i\tau} + 21493760e^{4\pi i\tau} + \cdots .$$

Theorem (Kronecker)

The maximal unramified abelian extension of K is $K(j(\tau))$.

All finite abelian extensions of K are contained in $K(j(\tau), \wp(\tau, z))$, where $\wp$ is the Weierstrass $\wp$ function, an elliptic function, and $z \in \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is a torsion point.

Real quadratic fields

Question

Can we construct abelian extensions of real quadratic fields using a similar method?

Issue

$j(a + b\sqrt{D})$ is undefined for $D > 0$.

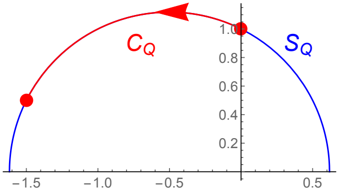
Overview of cycle integrals

Idea

Take the average of a modular function (such as j) along a geodesic path.

- Seems like a reasonable candidate for a real quadratic analogues of singular moduli
- Related (as we'll see) to coefficients of harmonic Maass forms
- ..but cycle integrals of the j-function seem to be transcendental. Maybe try other modular functions?

Definition of cycle integrals



Let $Q(x,y) = ax^2 + bxy + cy^2$, $\mathrm{disc}(Q) > 0$.

$$S_Q = \{\tau \in \mathbb{H} : a|\tau|^2 + b\,\mathrm{Re}(\tau) + c = 0\}.$$

Write $\mathrm{stab}_{\mathrm{SL}_2(\mathbb{Z})}(S_Q) = \langle g_Q \rangle$, $w \in S_Q$, C_Q a path from w to $g_Q w$.
The cycle integral of $f : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ for Q is

$$\int_{C_Q} f(z) \frac{dz}{Q(z,1)}.$$

Twisted races of cycle integrals

For a holomorphic modular function f, let

$$\mathrm{Tr}_D(f,\chi) = \sum_{Q\in\mathcal{Q}^+_{\mathrm{prim}}(D)} \chi(Q) \int_{C_Q} \frac{f(\tau)}{Q(\tau,1)} d\tau$$

for a character χ on $\mathcal{Q}^+_{\mathrm{prim}}(D)/\operatorname{SL}_2(\mathbb{Z})$.

Theorem (Duke, Imamoglu, and Tóth, 2011)

For χ a genus character, the values $\mathrm{Tr}_D(j,\chi)$ are coefficients of a weight 1/2 mock modular form.

Generalization of Duke-Imamoglu-Tóth

Theorem (Matsusaka, 2018)

Traces of polyharmonic modular functions are coefficients of the holomorphic part of half integral weight polyharmonic weak Maass forms.

Example

The function $f(z) = -\log(y|\eta(\tau)|^4)$ is a polyharmonic modular function which appears in the Kronecker Limit formula.

The twisted traces of $f(z)$ are coefficients of a polyharmonic weak Maass form of weight $\frac{1}{2}$.

Kronecker limit formula

For $\mathrm{Re}(k+2s) > 2$,

$$E_k(\tau, s) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{(m\tau+n)^k|m\tau+n|^{-2s}}.$$

Theorem (Kronecker limit formula)

For $\tau \in \mathbb{H}$

$$E_0(\tau, s) = \frac{2\pi}{s-1} + 2\pi(2\gamma_0 - \log 4 + \log(y|\eta(\tau)|^4)) + O(s-1)$$

for s in a neighborhood of 1. Here γ_0 is Euler's constant.

Note: For a fixed imaginary quadratic irrationality $\tau \in \mathbb{H}$, the function $E_0(\tau, s)$ is a partial ideal class zeta function for $\mathbb{Z}[\tau]$.

Kronecker limit formula for positive discriminants

Theorem (Hecke)

Let $D > 0$, $A \in \mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$, and let $\zeta(s, A) = \sum_{\mathfrak{a} \in A} \mathrm{Nm}(\mathfrak{a})^{-s}$ for $\mathrm{Re}(s) > 1$.

$$\zeta(s, A) = \frac{2D^{-1/2} \log \epsilon}{s-1} + \frac{2 \log \epsilon}{\sqrt{D}} \left(-\frac{1}{2} \log D + 2\gamma_0 \right) - \frac{1}{\sqrt{D}} \int_{C_Q} \log \left(y |\eta(\tau)|^4 \right) \frac{d\tau}{Q(\tau, 1)} + O(s-1).$$

Here ϵ is a fundamental unit and Q depends on A.

Theorem 3 (Beckwith and K, 2021+)

We obtain a generalization where A is a ray class of K and the Laurent coefficients are polyharmonic Maass forms for $\Gamma(N)$.

Definition

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let $r \in \frac{1}{2}\mathbb{N}$. A polyharmonic Maass form of weight k and depth r is a function $f : \mathbb{H} \to \mathbb{C}$ such that

1. $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$ for all $\gamma \in \Gamma$.
2. $\Delta_k^r(f) = 0$,
   where
   $$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$
3. There exists $c \in \mathbb{R}$ such that $f(x+iy) = O(y^c)$ as $y \to \infty$, and analogous conditions hold at the other cusps of Γ.

Remarks

- For $\xi_k = 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}$, we have $\Delta_k = \xi_{2-k} \circ \xi_k$.

  The definition makes sense for $r = \frac{1}{2}, \frac{3}{2}, \cdots$ by interpreting $\Delta_k^{\frac{1}{2}}$ as ξ_k.

- When $r = \frac{1}{2}$, these are holomorphic modular forms.
- When $r = 1$, these are harmonic Maass forms.
- When $r = 3/2$, these are sesquiharmonic Maass forms.
- When $r = 2$, these are biharmonic Maass forms.
- We let $V_k^{r,\Gamma}$ denote the space of such functions.

Level 1

Theorem (Lagarias and Rhoades, 2015)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The space $V_k^{r,\Gamma}$ is spanned by $S_k(\Gamma)$ and the first r Taylor coefficients of $E_k(\tau, s)$ at $s = 0$.

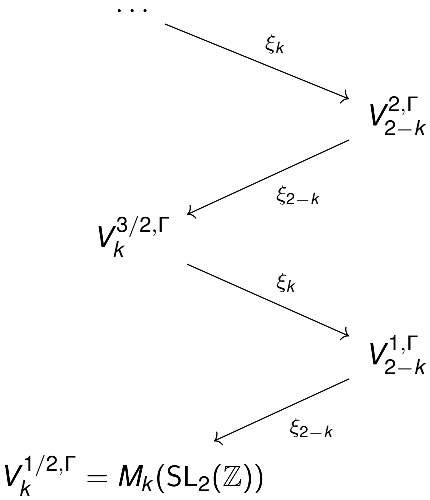
Example

$$E_0(\tau, s) = \sum_{n \geq 0} A_n(\tau) s^n.$$

$A_0(\tau) \in V_0^{1/2,\Gamma} = M_0(\Gamma)$.
$A_1(\tau) \in V_0^{3/2,\Gamma}$
$A_2(\tau) \in V_0^{5/2,\Gamma}$

Diagram

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For $k > 2$ and $k = 0$:



Generalizations

- Other eigenvalues: Andersen, Lagarias, Rhoades
- Polyharmonic weak Maass forms: Matsusaka
- Half integral weight polyharmonic Maass forms: Matsusaka
- Our work: polyharmonic Maass forms with respect to $\Gamma(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N}\}$.

Offset Eisenstein series

Definitions

Let $q_1, q_2 \in \mathbb{Q}$.

$$E^k_{q_1,q_2}(\tau, s) :=$$

$$\sum_{(m,n)\in\mathbb{Z}^2} \frac{y^s}{|(m+q_1)\tau + (n+q_2)|^{2s}\left((m+q_1)\tau + (n+q_2)\right)^k}$$

for $\mathrm{Re}(2s+k) > 2$.

$E^k_{q_1,q_2}$ have meromorphic continuation in the s-variable to $\mathbb{C}$.

Laurent expansion:

$$\sum_{j=-1}^{\infty} B^{k,j}_{q_1,q_2}(\tau)s^j := E^k_{q_1,q_2}(\tau, s).$$

Result on spaces of polyharmonic Maass forms

Theorem 4 (Beckwith and K, 2021+)

Let k be an integer not equal to 1. A basis for $V_k^{r,\Gamma(N)}$ is given in terms of

$$\left\{ B^{k,j}_{\frac{a_1}{N},\frac{a_2}{N}}(\tau) : (a_1, a_2) \in (\mathbb{Z}/N\mathbb{Z})^2, -1 \leq j \leq r+1 \right\}.$$

Example

For $k = 0$, if $q_1 = q_2 = \frac{1}{2}$, then

$$B^{0,r}_{1/2,1/2}(\tau) \in V_k^{r,\Gamma(2)}.$$

Diagram

For $k > 2$:

$$\cdots \xrightarrow{\xi_k} V^2_{2-k,\Gamma(N)}$$

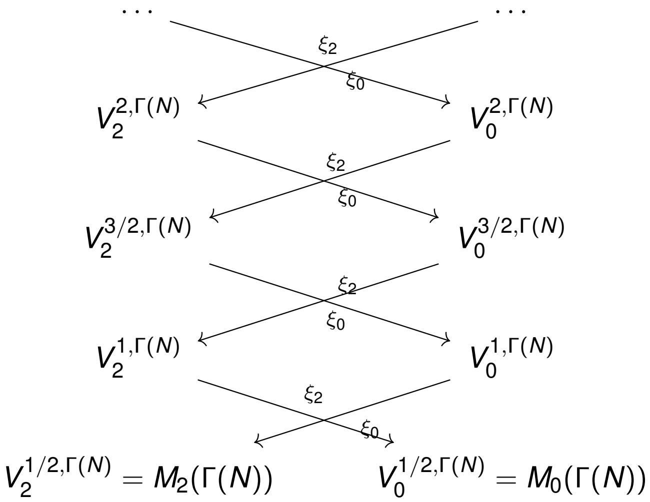
$$V^2_{2-k,\Gamma(N)} \xrightarrow{\xi_{2-k}} V_k^{3/2,\Gamma(N)}$$

$$V_k^{3/2,\Gamma(N)} \xrightarrow{\xi_k} V_{2-k}^{1,\Gamma(N)}$$

$$V_{2-k}^{1,\Gamma(N)} \xrightarrow{\xi_{2-k}} V_k^{1/2,\Gamma(N)} = M_k(\Gamma(N))$$

Diagram

For $k = 0$:



Hecke L-series

Definition

Let $(N, D) = 1$ and let χ be a character of a ray class group $\mathrm{Cl}^{+}_{(N)}(\mathcal{O}_D)$.

$$L(s, \chi) := \sum_{\mathfrak{a} \leq \mathcal{O}_D} \chi([\mathfrak{a}]) \, \mathrm{Nm}(\mathfrak{a})^{s}$$

for $\mathrm{Re}(s) > 1$ is the Hecke series for K with respect to χ.

- Products of these $L(s, \chi)$ are Dedekind zeta functions for totally real abelian extensions of K.
- The Stark conjectures predict that (in certain cases) $L''(0, \chi)$ is a quadratic form in logarithms of units of abelian extensions of K.

Hecke L-series

Let $\phi : \mathrm{Cl}_N^+(\mathcal{O}_D) \to \mathcal{Q}_{\mathrm{prim}}^{+,N}(D)/\Gamma_1(N)$ be as in Theorem 1.

Theorem 5 (Beckwith and K, 2021+)

If χ factors through $\mathrm{Cl}_N(\mathcal{O}_D)$, then

$$L(s,\chi) = \left(\frac{1}{2} \sum_{A \in \mathrm{Cl}_N^+(\mathcal{O})} \chi(A) \int_{C_\phi(A)} B^{0,2}_{\frac{1}{N},0}(\tau) \frac{d\tau}{\phi(A)(\tau,1)} \right) s^2 + O(s^3).$$

- The integrand is a biharmonic Maass form for $\Gamma_1(N)$.
- Proof idea: We use Hecke's method to compute $L(s,\chi)$ in terms of offset Eisenstein series.

Example

Consider $K = \mathbb{Q}(\sqrt{23})$, $N = 5$.

- $\mathrm{Cl}_5^+(K) \cong \mathbb{Z}/12\mathbb{Z}$.
- Let χ be an order 3 Hecke character of conductor $5\mathcal{O}_K$.
- This character defines a degree 3 abelian extension H_χ/K.
- $\mathrm{Gal}(H_\chi/\mathbb{Q}) \cong S_3$.

We can prove that $L(s,\chi) = \zeta_M(s)/\zeta(s)$ for a non-Galois cubic extension $M = \mathbb{Q}(\alpha)$.

Here, $\alpha^3 - 17\alpha^2 + 63\alpha + 1 = 0$ and $M(\sqrt{23}) = H_\chi$.

Example

Combining the class number formula with our result,

$$2\left(\log(\alpha)\log(\beta)-\log(-\alpha')\log(-\beta')\right)$$
$$=\frac{1}{2}\sum_{j=0}^{2}\chi(A_j)\int_{C_{Q_j}} B^{0,2}_{1/5,0}(\tau)\frac{d\tau}{Q(\tau,1)},$$

where

$$\alpha\approx 5.48872, \alpha'\approx -0.0158055 \text{ roots of } x^3-17x^2+63x+1=0,$$
$$\beta\approx 1.16151, \beta'\approx -74.1731 \text{ roots of } x^3+73x^2-87x+1=0.$$

The end.

Thank you all for your attention!

Questions?