# Complex equiangular lines and the Stark conjectures

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## Results

### Form of main result

Stark conjectures
+
Further hypotheses
$\implies$ SIC existence

This result is a reduction of:

- A problem in frame theory to a problem in number theory.
- A problem about complex numbers to a problem about real numbers.

Caveats:

- Valid for prime dimension $d \equiv 2 \pmod 3$ (general $d$ is work in progress).
- “Further hypotheses” are somewhat artificial.

# Complex equiangular lines

## Definition

A set $S \subset \mathbb{C}^d$ of unit norm vectors is equiangular with common angle $\arccos(\alpha)$ if the absolute value of the Hermitian inner product $|\langle v, w\rangle|^2 = \alpha$ when $v, w \in S$ and $v \neq w$.

## Example (d=2)

- $S = \left\{[1:0], [-\frac{1}{2} : \frac{\sqrt{3}}{2}], [-\frac{1}{2} : -\frac{\sqrt{3}}{2}]\right\}$ ...largest real set, coming from vertices of an equilateral triangle on $\mathbb{P}^1(\mathbb{R}) \cong S^1$. $\alpha = \frac{1}{2}$.
- $S = \{[1 : \frac{1+i}{1+\sqrt{3}}], [1 : \frac{-1-i}{1+\sqrt{3}}], [\frac{1+i}{1+\sqrt{3}} : 1], [\frac{-1-i}{1+\sqrt{3}} : 1]\}$ ...largest complex set, coming from vertices of a regular tetrahedron on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. $\alpha = \frac{1}{\sqrt{3}}$.

## Real vs complex cases

- In $\mathbb{R}^d$, the size of a set of equiangular lines is bounded above by $\binom{d+1}{2} = \dim_{\mathbb{R}} \mathrm{Sym}^2(\mathbb{R}^d)$. If this maximum is achieved, the common angle is given by $\cos\theta = \alpha = \frac{1}{\sqrt{d+2}}$.
- However, this theoretical maximum is conjectured to be achieved only in dimensions $d = 1, 2, 3, 7, 23$.
- In $\mathbb{C}^d$, the size of a set of equiangular lines is bounded above by $d^2 = \dim_{\mathbb{R}} \mathrm{Herm}(\mathbb{C}^d)$. If this maximum is achieved, the common angle is given by $\cos\theta = \alpha = \frac{1}{\sqrt{d+1}}$.
- This theoretical maximum is conjectured to be achieved in every dimension.

## SICs

### Definition

A SIC (SIC-POVM; symmetric informationally complete positive operator-valued measure) is (a set of quantum measurements equivalent to) a set of $d^2$ equiangular lines in $\mathbb{C}^d$.

### Definition

The overlaps of a SIC $\{v_1, \ldots, v_{d^2}\}$ are the $d^4 - d^2$ complex numbers $\langle v_i, v_j \rangle$ of absolute value $\frac{1}{\sqrt{d+1}}$ $(i \neq j)$.

The complex numbers $\sqrt{d+1}\langle v_i, v_j \rangle$ are called overlap phases.

# Weyl-Heisenberg SICs

All but one of the known SICs are Weyl-Heisenberg (covariant) SICs: The orbit of a single fiducial vector $v$ under the discrete Weyl-Heisenberg group $\mathrm{H}(d) = \langle X, Z \rangle$.

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_d & 0 & \cdots & 0 \\ 0 & 0 & \zeta_d^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_d^{d-1} \end{pmatrix}.$$

The group $\mathrm{H}(d)$ is image of the discrete Heisenberg group of the ring $\mathbb{Z}/d\mathbb{Z}$ under the Weyl algebra representation.

## Weyl-Heisenberg SICs

The Weyl-Heisenberg group has order $d^3$, but its center has order $d$ and acts by scalars. A Weyl-Heisenberg SIC has the form

$$S = \{D_{m,n}v : 0 \le m, n \le d-1\}$$

$$D_{m,n} = e^{\frac{(d+1)mn\pi i}{d}} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_d & 0 & \cdots & 0 \\ 0 & 0 & \zeta_d^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_d^{d-1} \end{pmatrix}^n$$

Up to multiplication by roots of unity, the overlaps of a Weyl-Heisenberg SIC are the $d^2 - 1$ numbers $\langle v, D_{m,n}v \rangle$ with $(m, n) \neq (0, 0)$.

## (When) Do SICs exist?

**Conjecture (Zauner 1999)**

Weyl-Heisenberg SICs exist in every dimension $d$.

- Zauner's conjecture is only known for finitely many $d$.
- Exact algebraic solutions in dimensions 1–21, 23, 24, 28, 30, 31, 35, 37, 39, 43, 48, 53, 124, 195, and 323. Numerical (probable) solutions in every dimension up to 151 and several other dimensions up to 844.
- Surprising observation: in known examples, the field of definition of Weyl-Heisenberg SICs in dimension $d \geq 4$ is an abelian extension of $K = \mathbb{Q}(\sqrt{(d+1)(d-3)})$, often a particular ray class field $L_{(d)\infty_1}$ (Appleby, Flammia, McConnell, and Yard; 2016).

Let $K$ be a number field and $\mathcal{O}_K$ its ring of integers. Let $\mathfrak{c}$ be a ideal in $\mathcal{O}_K$, and let $S$ be a subset of the real embeddings of $K$.

**Definition (Ray class group modulo $\mathfrak{c}, S$)**

$$\mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K) = \frac{\{\text{fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K \text{ s.t. } a \equiv 1 \pmod{\mathfrak{c}} \text{ and } \rho(a) > 0 \text{ for } \rho \in S\}}$$

Class field theory associates to $\mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$ a ray class field $L_{\mathfrak{c},S}$, an abelian extension of $K$ with Galois group $\mathrm{Gal}(L_{\mathfrak{c},S}/K) = \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$. Varying $\mathfrak{c}$ and $S$, the ray class fields are cofinal among all abelian extensions of $K$.

## Hilbert's 12 problem and the Stark conjectures

- 12th problem asks for an "Extension of Kronecker's Theorem on Abelian Fields to any Algebraic Realm of Rationality."
- Kronecker's Theorem (Kronecker-Weber theorem) says that the abelian extensions of $\mathbb{Q}$ are generated by the values of $e(z) = e^{2\pi iz}$ at rational values of $z$.
- Given any base field ("realm of rationality"), Hilbert wanted "analytic functions" that play the role of $e(z)$.
- Harold Stark conjectured in a series of papers (1971–1980) that $\exp(cZ(1))$, for certain linear combinations $Z(s)$ of Hecke $L$-functions of $K$, generate abelian extensions of $K$.

# $L$-functions at $s = 1$: rational example

The following formula can be proved using calculus. Try it!

**Example**

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + - - + \cdots = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right)$$

The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. The right-hand side involves $\varepsilon = 1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.

# $L$-functions at $s = 1$: imaginary quadratic example

The following formula is proved using the theory of complex multiplication for elliptic curves. The notation $e(z) := e^{2\pi i z}$.

**Example**

$$\sum\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{e(m/5) - e(2m/5)}{m^2 + mn + n^2} = \frac{2\pi}{\sqrt{3}} \log\left(\varepsilon^{1/5}\right)$$

where $\varepsilon = 29 + 12\sqrt{5} + 2\sqrt{6(65 + 29\sqrt{5})}$.

The left-hand side is a linear combination of Hecke L-values at $s = 1$ for $\mathbb{Q}(\sqrt{-3})$. The right-hand side involves an algebraic unit $\varepsilon$ in the ray class field modulo (5) for $\mathbb{Q}(\sqrt{-3})$.
This example is related to the 5-torsion points of the CM elliptic curve $y^2 = x^3 + 1$.

The following formula is an open conjecture!

**Example**

$$\sum_{m=1}^{\infty} \sum_{\substack{n\in\mathbb{Z} \\ -\frac{5}{3}m\le n<\frac{5}{3}m}} \frac{e(4m/5) - e(m/5)}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log(\varepsilon),$$

where $\varepsilon \approx 3.890861714$ is a root of the polynomial equation

$$x^8 - (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 + (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 - (8+5\sqrt{3})x + 1 = 0.$$

The number $\varepsilon$ is an algebraic unit in the ray class field of $\mathbb{Q}(\sqrt{3})$ modulo $5\infty_2$. This conjecture is part of the Stark conjectures.

## Zeta functions associated to ray classes

**Definition**

For $A \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$, the associated zeta function is

$$\zeta(s, A) = \sum_{\substack{\mathfrak{a} \leq \mathcal{O}_K \\ \mathfrak{a} \in A}} N(\mathfrak{a})^{-s}.$$

Let $R \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$ be the ideal class

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } \rho(a) > 0 \text{ for } \rho \in S\}.$$

**Definition**

For $A \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$, the associated differenced zeta function is

$$Z_A(s) = \zeta(s, A) - \zeta(s, RA).$$

**Conjecture (Stark, 1976)**

Setup:

- Let $K$ be a real quadratic number field.
- Consider $0 \neq \mathfrak{c} \leq \mathcal{O}_K$ with the property that, if $\varepsilon \in \mathcal{O}_K^\times$ and $\varepsilon \equiv 1 \pmod{\mathfrak{c}}$, then one of $\varepsilon$ or $-\varepsilon$ is totally positive.
- Let $A$ be a ray ideal class in $\mathrm{Cl}_{\mathfrak{c}\infty_2}(\mathcal{O}_K)$.
- Let $L_{\mathfrak{c}\infty_j}$ be the ray class field of $K$ modulo $\mathfrak{c}\infty_j$.
- Let $\rho_j$ be the real embedding of $L_{\mathfrak{c}\infty_j}$ associated to $\infty_j$.

Then,

**(1)** $Z_A'(0) = \log(\rho_1(\varepsilon_A))$ for a unit $\varepsilon_A \in L_{\mathfrak{c}\infty_2}$.

**(2)** The units $\varepsilon_A$ are compatible with the isomorphism $\mathrm{Art} : \mathrm{Cl}_{\mathfrak{c}\infty_2}(\mathcal{O}_K) \to \mathrm{Gal}(L_{\mathfrak{c}\infty_2}/K)$. Specifically, $\varepsilon_A = \varepsilon_I^{\mathrm{Art}(A)}$.

# Example

- Let $K = \mathbb{Q}(\sqrt{3})$, so $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$, and let $\mathfrak{c} = 5\mathcal{O}_K$.
- The ray class group $\mathrm{Cl}_{\mathfrak{c}\infty_2} \cong \mathbb{Z}/8\mathbb{Z}$. Let $I$ be the identity.
- We can calculate $Z_I'(0) \approx 1.3586306534$ and $\exp(Z_I'(0)) \approx 3.8908617139$—apparently the root of a degree 8 polynomial.

$$
\begin{aligned}
x^8 &- (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 \\
&+ (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 \\
&- (8+5\sqrt{3})x + 1 = 0.
\end{aligned}
$$

## An observation when $d = 5$

### Observation

For an appropriate choice of fiducial vector, the squares of the overlap phases (each having multiplicity 3) of a Weyl-Heisenberg SIC in dimension $d = 5$ are the roots of the polynomial

$$x^8 - (8 - 5\sqrt{3})x^7 + (53 - 30\sqrt{3})x^6 - (156 - 90\sqrt{3})x^5 + (225 - 130\sqrt{3})x^4 - (156 - 90\sqrt{3})x^3 + (53 - 30\sqrt{3})x^2 - (8 - 5\sqrt{3})x + 1 = 0.$$

That is...the same polynomial on the previous slide, except with $\sqrt{3}$ replaced by $-\sqrt{3}$.

Might this observation generalise?

## Quasi-SICs

Relax (complexify) the conditions defining a SIC.

### Definition

A (unit norm Weyl-Heisenberg) quasi-SIC is a pair $(v, w) \in \mathbb{C}^d$ such that

$$(v^\top D_{mn} w)(v^\top D_{mn}^{-1} w) = \begin{cases} \frac{1}{d+1} & \text{if } (m,n) \neq (0,0), \\ 1 & \text{if } (m,n) = (0,0). \end{cases}$$

The set of quasi-SICs forms an algebraic variety. If we mod out by phase $(v, w) \sim (\alpha v, \alpha^{-1} w)$, guess that we get a zero-dimensional variety (for $d \neq 3$).

## SICs and ghost SICs

A quasi-SIC is a SIC if and only if $w = \overline{v}$.

**Definition (Appleby)**

A ghost SIC is a quasi-SIC with the property that $w = P\overline{v}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

- The overlaps of a SIC are on the circle of radius $\frac{1}{\sqrt{d+1}}$.
- The overlaps of a ghost SIC are on the real line.

## Conjecture 1 (K; existence of special units)

Let $d \equiv 2 \pmod 3$ be an odd prime. Let $\Delta = (d+1)(d-3)$ and $K = \mathbb{Q}(\sqrt{\Delta})$. With indices $m, n \in \mathbb{Z}/d\mathbb{Z}$, let

$$A_{m,n} = \{\alpha\mathcal{O}_K : \alpha \equiv m + n\sqrt{\Delta} \pmod d \text{ and } \rho_2(\alpha) > 0\} \in \mathrm{Cl}_{(d)\infty_2}.$$

Then, there is a real algebraic unit $\alpha$ such that the ray class field $L_{(d)\infty_2} = K(\alpha)$ and $\alpha_{m,n} := \alpha^{\mathrm{Art}(A_{m,n})}$ satisfy:

- The roots of $(d+1)x^2 = \alpha_{m,n}$ are in $L_{(d)\infty_2}$.
- A particular choice of roots $\nu_{m,n} = \pm\sqrt{\frac{\alpha_{m,n}}{d+1}}$ (defined by a congruence condition) are the overlaps of a ghost SIC.

# Conjecture 2 – analytic construction of ghost SICs

## Conjecture 2 (K; find special units as Stark units)

A unit $\alpha$ satisfying Conjecture 1 and its Galois conjugates over $K$ may be constructed as Stark units

$$\alpha^{\mathrm{Art}(A)} = \exp\left(Z_A'(0)\right),$$

for all $A \in \mathrm{Cl}_{(d)\infty_2}$.

As before, the differenced ray class zeta function $Z_A(s)$ is defined as

$$Z_A(s) = \left(\sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}\right) - \left(\sum_{\mathfrak{a}\in RA} N(\mathfrak{a})^{-s}\right),$$

where $R = \{a\mathcal{O}_K : a \equiv -1 \pmod{d} \text{ and } \rho_2(a) > 0\}$.

# Results

**Theorem (K)**

Let $d$ be an odd prime such that $d \equiv 2 \pmod{3}$. In dimension $d$,

$$\text{Conjecture 1} \implies \text{SIC existence.}$$

- The Stark unit construction of Conjecture 2 works (numerically) at least for $d = 5, 11, 17$, and 23 ($d = 53$ has been "spot-checked").
- After finding the corresponding exact units by lattice basis reduction, we provide the first exact construction of a SIC in dimension 23.

# Work in progress

Generalise the theorem to general $d \geq 4$.

**Theorem (Appleby, Flammia, K, 2020+; in progress)**

Under the "class field hypothesis", there is a (non-canonical) one-to-one bijection between SICs and ghost SICs, given by a choice of Galois automorphism $\sigma \in \mathrm{Gal}(L_{d\infty_1\infty_2}/\mathbb{Q})$ such that $\sigma(\sqrt{(d+1)(d-3)}) = -\sqrt{(d+1)(d-3)}$.

## Work in progress

- Generalise my construction to arbitrary $d$ / arbitrary Clifford group orbits (K, Lagarias; Appleby, Flammia, K).
- Scale up numerical construction of SICs from $L$-functions (Appleby, Flammia, K).

## Long-term goals

Zauner's conjecture remains open.

- Constructive vs non-constructive approaches
- €2020 prize (Golden KCIK award) by the National Quantum Information Centre in Poland (search "Five open problems in quantum information" on arXiv)

Another question: Is there a generalisation of the SIC condition that applies to Stark units over $\mathbb{Q}(\Delta)$ when $\Delta \neq (d+1)(d-3)$, or over more general number fields?

# Thank you!

Thank you for listening, and thank you to the organisers!

Kopp, Gene. SIC-POVMs and the Stark conjectures. Preprint available at arxiv:1807.05877. To appear in *Int. Math. Res. Notices*, rnz153.