A KRONECKER LIMIT FORMULA FOR INDEFINITE ZETA FUNCTIONS

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Abstract. Indefinite zeta function are Mellin transforms of indefinite theta function (á la Zwegers) and specialise to differenced ray class zeta functions of real quadratic fields. We prove a Kronecker limit formula for indefinite zeta functions in dimension $g = 2$ at $s = 1$. This formula specialises to a new analytic formula for presumptive Stark units.

1. Introduction

The Hecke L-value $L_K(1, \chi)$ contains arithmetic information that is not well-understood in general. The abelian Stark conjectures predict that this value is an algebraic number times a regulator Reg_χ, which is a determinant of a matrix of linear forms in logarithms of algebraic units in a particular abelian extension of the number field K [10, 11, 12]. This conjecture is known when the base field K is equal to $\mathbb{Q}$ or an imaginary quadratic field, but not (for instance) when K is a real quadratic field.

The Stark conjectures are most precisely formulated in the rank 1 case—that is, when $L_K(s, \chi)$ vanishes to order 1 at $s = 0$. The regulator Reg_χ in that case is a determinant of a 1×1 matrix. The Stark conjectures are most succinctly written as a statement about the ray class zeta function (of a ray ideal class A)

$$\zeta(s, A) = \zeta_K(s, A) = \sum_{\mathfrak{a} \in A} N(A)^{-s} \tag{1.1}$$

rather that as a statement about the Hecke L-function

$$L_K(s, \chi) = \sum_{A} \zeta(s, A). \tag{1.2}$$

The rank 1 abelian Stark conjectures give a partial answer to Hilbert's 12th Problem, which asked for explicit generators for the abelian extensions of a number field in terms of special values of transcendental functions. Computationally, the Stark conjectures are used to calculate class fields in the computer algebra systems Magma and PARI/GP.

For imaginary quadratic fields, Stark proved his conjectures using Kronecker's first and second limit formulas together with the theory of singular moduli [10]. The Kronecker limit formulas give the constant Laurent series coefficient at $s = 1$ for families of Dirichlet series continuously interpolating the ray class zeta functions $\zeta(s, A)$—namely, standard and twisted real analytic Eisenstein series.

Analogous formulas applicable to real quadratic fields were developed by Hecke, Herglotz, and Zagier (in analogy with the first limit formula) and by Shintani (in analogy with the

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second). As in the imaginary quadratic case, these formulas are obtained by continuously interpolating between ray class zeta functions using a larger family of functions. Hecke's formula uses cycle integrals of real analytic Eisenstein series, whereas the formulas of Herglotz [3], Zagier [13], and Shintani [7, 8] use partial zeta functions defined by summing over a cone.

This paper gives a new real quadratic analogue of Kronecker's second limit formula based on a new interpolation between ray class zeta functions. The interpolation is by **indefinite zeta functions** introduced in [5], which are Mellin transforms of Zwegers's nonholomorphic indefinite theta functions.

Our setup also allows us to prove a Kronecker limit formula for definite zeta functions that generalises Kronecker's second limit formula.

1.1. **Kronecker's first limit formula and imaginary quadratic L-values.** The real analytic Eisenstein series

$$E(\tau,s) := \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{\operatorname{Im}(\tau)^s}{|m\tau+n|^{2s}} \tag{1.3}$$

is closely related to the zeta function of an imaginary quadratic ideal class

$$\zeta(s,A) := \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}. \tag{1.4}$$

Specifically, if A is an ideal class of the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K, and we choose any $\mathfrak{b}\in A^{-1}$ such that $\mathfrak{b}\cap\mathbb{Q}=\mathbb{Z}$ and write $\mathfrak{b}=\mathbb{Z}+\tau\mathbb{Z}$ for $\operatorname{Im}(\tau)>0$, then

$$N(\mathfrak{b})^{-s}\zeta(s,A) = \sum_{\mathfrak{a}\in A} N(\mathfrak{b}\mathfrak{a})^{-s} \tag{1.5}$$

$$= \sum_{\alpha\in\mathfrak{b}/\mathcal{O}_K^\times} N(\alpha)^{-s} \tag{1.6}$$

$$= \left|\mathcal{O}_K^\times\right| \sum_{\alpha\in\mathfrak{b}} N(\alpha)^{-s} \tag{1.7}$$

$$= \left|\mathcal{O}_K^\times\right| \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} |m\tau+n|^{-s} \tag{1.8}$$

$$= \frac{\left|\mathcal{O}_K^\times\right|}{\operatorname{Im}(\tau)^s} E(\tau,s). \tag{1.9}$$

Write $\tau = x+yi$ for real numbers x,y. The real analytic Eisenstein series has a Fourier series in x (see, e.g., [2], chapter 1, pages 67–69). We write it using the completed Eisenstein series $E^*(\tau,s) := \frac{1}{2}\pi^{-s}\Gamma(s)E(\tau,s)$ and the completed Riemann zeta function $\hat{\zeta}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

$$E^*(\tau,s) = \hat{\zeta}(2s)y^s + \hat{\zeta}(2s-1)y^{1-s} \tag{1.10}$$

$$+ 2\sqrt{y}\sum_{m\in\mathbb{Z}\setminus\{0\}} |m|^{s-\frac{1}{2}}\sigma_{1-2s}(|m|)K_{s-\frac{1}{2}}(2\pi|m|y)e(mx). \tag{1.11}$$

By sending $s \to 1$ in this Fourier expansion, we obtain the **first Kronecker limit formula**.

$$\lim_{s\to 1}\left(E^*(\tau,s)-\frac{1}{2(s-1)}\right) = -2\log|\eta(\tau)| + \lim_{s\to 1}\left(\hat{\zeta}(2s-1)y^{1-s}-\frac{1}{2(s-1)}\right) \tag{1.12}$$

$$= -2\log|\eta(\tau)| + \frac{1}{2}\left(\gamma - \log(4\pi y)\right). \tag{1.13}$$

Here, $\eta(\tau)$ is the Dedekind eta function $\eta(\tau) = e(\tau/24)\prod_{d=1}^{\infty}(1-e(d\tau))$, a modular form of weight $\frac{1}{2}$, and γ is the Euler-Mascheroni constant. A detailed proof of the first Kronecker limit formula may be found in [6], chapter 20, pages 273–275.

From this formula, we can obtain the constant term of the Taylor expansion of $\zeta(s,A)$ at $s=1$, or, equivalently by the functional equation, the value of $\zeta'(s,A)$ at $s=0$. Using results from the theory of elliptic curves with complex multiplication, one can show that integral linear combinations of the $\zeta'(0,A)$ whose coefficients sum to zero are logarithms of algebraic numbers (as they're logarithms of absolute values of modular functions evaluated at moduli of CM elliptic curves). Moreover, one may show that these algebraic numbers are algebraic units in an appropriate class field. Stark does so in the first paper of his series [10].

1.2. **Notational conventions.** We list some notational conventions used in the paper that may need clarification.

- $e(z) := \exp(2\pi i z)$ is the complex exponential, and this notation is used for $z \in \mathbb{C}$ not necessarily real.
- $\mathcal{H} = \{\tau : \operatorname{Im}\tau > 0\}$ is the complex upper half-plane.
- Non-transposed vectors $v \in \mathbb{C}^g$ are always column vectors; the transpose $v^\top$ is a row vector.
- If M is a $g \times g$ matrix, then $M^\top$ is its transpose, and (when M is invertible) $M^{-\top}$ is a shorthand for $(M^{-1})^\top$.
- $Q_M(v)$ denotes the quadratic form $Q_M(v) = \frac{1}{2}v^\top M v$, where M is a $g \times g$ matrix, and v is a $g \times 1$ column vector.
- $f(c)|_{c=c_1}^{c_2} = f(c_2) - f(c_1)$, where f is any function taking values in an additive group.
- If $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in \mathbb{C}^2$ and f is a function of $\mathbb{C}^2$, we may write $f(v) = f\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ rather than $f\left(\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)\right)$.
- We will often need to express $\Omega = iM + N$ where M, N are real $g \times g$ symmetric matrices; N and M will always have real entries even when we do not say so explicitly.

We will use complex logarithms throughout this paper. If $f(\tau)$ is any nonvanishing holomorphic function on the upper half plane $\mathcal{H}$, there is some holomorphic function $(\operatorname{Log} f)(\tau)$ such that $\exp((\operatorname{Log} f)(\tau)) = f(\tau)$, because $\mathcal{H}$ is simply connected. Specifying a single value (or the limit as τ approaches some element of $\mathbb{R} \cup \{\infty\}$) specifies $\operatorname{Log} f$ uniquely. It won't necessarily be true that $(\operatorname{Log} f)(\tau) = \log(f(\tau))$.

We also recall the definition of the Siegel intermediate half-space, which was defined in [5].

Definition 1.1. For $0 \le k \le g$, we define the **Siegel intermediate half-space** of genus g and index k to be

$$\mathcal{H}_g^{(k)} = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^\top \text{ and } \operatorname{Im}(\Omega) \text{ has signature } (g-k,k)\}. \tag{1.14}$$

The $\mathcal{H}_g^{(k)}$ are the open orbits of the action of $\mathbf{Sp}_{2g}(\mathbb{R})$ by fractional linear transformations on the space of complex symmetric matrices. In particular, $\mathcal{H}_g^{(0)}$ is the usual Siegel upper half-space.

1.3. **Definite zeta functions and the second limit formula.** We will now discuss the second limit formula and our results generalizing it, which are proved in section 2. A direct proof of the second limit formula may be found in [9].

We define the definite zeta function using a Mellin transform of the Riemann theta function $\Theta_{p,q}(\Omega)$ with real characteristics.

Definition 1.2. Let $\Omega = N + iM \in \mathcal{H}_g^{(0)}$ and $p, q \in \mathbb{R}^g$. For $q \notin \mathbb{Z}^g$ and $\mathrm{Re}(s) > 1$, define the **completed definite zeta function**

$$\hat{\zeta}_{p,q}(\Omega, s) = \int_0^\infty \Theta_{p,q}(t\Omega) t^s \frac{dt}{t}, \tag{1.15}$$

where

$$\Theta_{p,q}(\Omega) = \sum_{n \in \mathbb{Z}^g + q} e\left(n^\top \Omega n + p^\top n\right). \tag{1.16}$$

For $q \in \mathbb{Z}^g$ and $\mathrm{Re}(s) > 1$, define the completed definite zeta function by

$$\hat{\zeta}_{p,q}(\Omega, s) = \int_0^\infty \left(\Theta_{p,q}(t\Omega) - 1\right) t^s \frac{dt}{t}. \tag{1.17}$$

By taking the Mellin transform term-by-term, we see that the definite zeta function has a Dirichlet series. For $\mathrm{Re}(s) > 0$,

$$\hat{\zeta}_{p,q}(\Omega, s) = (2\pi)^{-s}\Gamma(s) \sum_{\substack{n \in \mathbb{Z}^g + q \\ n \neq 0}} \frac{e(p^\top n)}{(-in^\top \Omega n)^s}. \tag{1.18}$$

Here, the s-power function $z \mapsto z^s$ is defined with a branch cut on the negative real axis; note that $\mathrm{Re}(-in^\top \Omega n) > 0$ for all $n \neq 0$.

In the special case when $\Omega = iM$ for a positive definite real symmetric matrix M, $-in^\top \Omega n = 2Q_M(n)$ is positive definite a real quadratic form. The indefinite zeta function is then a twisted (by $e(p^\top n)$) offset (by q) Epstein zeta function.

Further specializing to $g = 2$, normalizing M to have determinant 1, and writing $M = \frac{1}{\mathrm{Im}(\tau)} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$, we obtain a twisted offset real analytic Eisenstein series.

$$\hat{\zeta}_{p,q}(\Omega, s) = (2\pi)^{-s}\Gamma(s) \sum_{\substack{n = \left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right) \in \mathbb{Z}^2 + q \\ n \neq 0}} \frac{e(p^\top n) \left(\mathrm{Im}(\tau)\right)^s}{\left|n_1 + n_2\tau\right|^{2s}}. \tag{1.19}$$

This specialisation is discussed further in section 4.2 of [5].

The classical second Kronecker limit formula for twisted real analytic Eisenstein series, stated in our notation, is as follows.

Theorem 1.3 (Second Kronecker limit formula). *Let* $p = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ *and* $\Omega = iM = \frac{i}{\mathrm{Im}(\tau)} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ \mathrm{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ *for* $\tau \in \mathcal{H}$. *Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = -2 \log \left| u^{p_1^2/2+1/12} \left(v^{1/2} - v^{-1/2}\right) \prod_{d=1}^{\infty} \left(1 - u^d v\right) \left(1 - u^d v^{-1}\right) \right| \tag{1.20}$$

where $u = e(\tau)$ *and* $v = e(p_2 - p_1\tau)$. *This formula may be written more compactly as*

$$\hat{\zeta}_{p,0}(\Omega, 1) = -2 \log \left| \frac{\vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(\tau)}{\eta(\tau)} \right|. \tag{1.21}$$

This paper generalises Theorem 1.3 to arbitrary $\Omega \in \mathcal{H}_2^{(0)}$. The following theorem will be proved in section 2.

Theorem 1.4 (Generalised second Kronecker limit formula). *Let* $p = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \in \mathbb{R}^2$ *with* $0 \leq p_1, p_2 < 1$, *and let* $\Omega = N + iM \in \mathcal{H}_2^{(0)}$. *Let* $z = \tau_1$ *and* $z = \tau_2$ *be the solutions of* $Q_\Omega \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = 0$ *in the upper and lower half-planes, respectively. Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = \frac{-1}{\sqrt{\det(-i\Omega)}} \left((\log f_p)(\tau_1) + (\log f_p)(-\tau_2) \right), \tag{1.22}$$

where the function $f_p : \mathcal{H} \to \mathbb{C}$ *may be written either of the following ways,*

$$\begin{aligned} f_p(\tau) &= e\left(-\frac{p_2}{2}\right) u_\tau^{p_1^2/2+1/12} \left(v_\tau^{1/2} - v_\tau^{-1/2}\right) \prod_{d=1}^{\infty} \left(1 - u_\tau^d v_\tau\right) \left(1 - u_\tau^d v_\tau^{-1}\right) && (1.23) \\ &= \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right) \vartheta_{\frac{1}{2}+p_2, \frac{1}{2}-p_1}(\tau)}{\eta(\tau)}, && (1.24) \end{aligned}$$

where $u_\tau = e(\tau)$, $v_\tau = e(p_2 - p_1\tau)$, ϑ *is the Jacobi theta function, and* η *is the Dedekind eta function. Here* $\mathrm{Log}\, f_p$ *is the branch satisfying*

$$(\mathrm{Log}\, f_p)(\tau) \sim \pi i \left(p_1^2 - p_1 + \frac{1}{6}\right) \tau \text{ as } \tau \to i\infty. \tag{1.25}$$

The completed definite zeta function has an analytic continuation and satisfies a functional equation, which we prove in section 2.4.

Theorem 1.5. *The function* $\hat{\zeta}_{p,q}(\Omega, s)$ *may be analytically continued to an entire function on* $\mathbb{C}$. *It satisfies the functional equation*

$$\hat{\zeta}_{p,q}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,p}\left(-\Omega^{-1}, s\right). \tag{1.26}$$

From Theorem 1.4 and Theorem 1.5, we can deduce a Kronecker limit formula for the completed definite zeta function at $s = 0$ in the case when $q = 0$. A full proof is given is section 2.4.

Theorem 1.6 (Generalised second Kronecker limit formula at $s = 0$). *Let* $q = \left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ *with* $0 \leq q_1, q_2 < 1$, *and let* $\Omega = N + iM \in \mathcal{H}_2^{(0)}$. *Let* $z = \tau_1$ *and* $z = \tau_2$ *be the solutions of* $Q_\Omega \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = 0$ *in the upper and lower half-planes, respectively. Then,*

$$\hat{\zeta}_{p,0}(\Omega, 1) = -\left((\mathrm{Log}\, g_q)(\tau_1) + (\mathrm{Log}\, g_q)(-\tau_2) \right), \tag{1.27}$$

where the function $g_q : \mathcal{H} \to \mathbb{C}$ is given by

$$g_q(\tau) = \frac{\vartheta_{\frac{1}{2}-q_1,\frac{3}{2}-q_2}(\tau)}{\eta(\tau)}. \tag{1.28}$$

For completeness, we give the specialisation of Theorem 1.6 to the case "offset real analytic Eisenstein series", as a classical Kronecker second limit formula at $s = 0$.

Proposition 1.7. *Let $q = \left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right) \in \mathbb{R}^2$ and $\Omega = iM = \frac{i}{\operatorname{Im}(\tau)}\begin{pmatrix} 1 & \operatorname{Re}(\tau) \\ \operatorname{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ for $\tau \in \mathcal{H}$. Then,*

$$\hat{\zeta}_{0,q}(\Omega, 0) = -2\log\left|\frac{\vartheta_{\frac{1}{2}-q_1,\frac{3}{2}-q_2}(\tau)}{\eta(\tau)}\right|. \tag{1.29}$$

Proof. Follows from Theorem 1.6 by specialisation of the variables. □

1.4. **Indefinite theta and zeta functions.**

Definition 1.8. For any complex number α, define the function

$$\mathcal{E}(\alpha) := \int_0^\alpha e^{-\pi u^2}\, du, \tag{1.30}$$

where the integral runs along any contour from 0 to α.

Definition 1.9. Let $\Omega = N + iM$ be a complex symmetric matrix whose imaginary part has signature $(g-1, 1)$; that is, $\Omega \in \mathcal{H}_g^{(1)}$. Define the **indefinite theta function**

$$\Theta^{c_1,c_2}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega) c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right), \tag{1.31}$$

where $z \in \mathbb{C}^g$, $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, and $\overline{c_2}^\top M c_2 < 0$.

Zwegers's theta function is defined for real c_j when N is a scalar multiple of M. More precisely, if M is real symmetric matrix of signature $(g-1, 1)$, $\tau \in \mathcal{H}$, and $c_1, c_2 \in \mathbb{R}^g$, then $\Theta^{c_1,c_2}(Mz, \tau M)$ is equal up to an exponential factor to the function $\vartheta_M^{c_1,c_2}(z, \tau)$ introduced by Zwegers on page 27 of [14].

Definition 1.10. Let $\Omega = N + iM \in \mathcal{H}_g^{(1)}$. Define the **indefinite theta null with characteristics** $p, q \in \mathbb{R}^g$:

$$\Theta_{p,q}^{c_1,c_2}(\Omega) = e\left(\frac{1}{2}q^\top \Omega q + p^\top q\right)\Theta^{c_1,c_2}(p + \Omega q; \Omega). \tag{1.32}$$

where $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, and $\overline{c_2}^\top M c_2 < 0$.

We define the indefinite zeta function using a Mellin transform of the indefinite theta function with characteristics.

Definition 1.11. Let $\Omega = N + iM \in \mathcal{H}_g^{(1)}$. The **completed indefinite zeta function** is

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \int_0^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega) t^s \frac{dt}{t}, \tag{1.33}$$

where $p, q \in \mathbb{R}^g$, and $c_1, c_2 \in \mathbb{C}^g$ are parameters satisfying $\overline{c_1}^\top M c_1 < 0$ and $\overline{c_2}^\top M c_2 < 0$.

The completed indefinite zeta function has an analytic continuation and satisfies a functional equation, which is Theorem 1.1 of [5].

Theorem 1.12 (Analytic continuation and functional equation)**.** *The function* $\hat{\zeta}_{a,b}^{c_1,c_2}(\Omega, s)$ *may be analytically continued to an entire function on* $\mathbb{C}$*. It satisfies the functional equation*

$$\hat{\zeta}_{p,q}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}, s\right). \tag{1.34}$$

1.5. **Indefinite zeta functions, real quadratic fields, and Stark units.** Just as definite zeta functions specialise to ray class zeta functions of imaginary quadratic fields, indefinite zeta functions specialise to **differenced ray class zeta functions** of real quadratic fields. The full details of this specialisation are given in section 7 of [5].

Definition 1.13 (Ray class zeta function)**.** Let K be any number field and $\mathfrak{c}$ an ideal of the maximal order $\mathcal{O}_K$. Let S be a subset of the real places of K (i.e., the embeddings $K \hookrightarrow \mathbb{R}$). Let A be a ray ideal class modulo $\mathfrak{c}S$, that is, an element of the group

$$\mathrm{Cl}_{\mathfrak{c}S}(\mathcal{O}_K) := \frac{\{\text{nonzero fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}}. \tag{1.35}$$

Define the *zeta function of* A to be

$$\zeta(s, A) = \sum_{\mathfrak{a} \in A} N(\mathfrak{a})^{-s}. \tag{1.36}$$

This function has a simple pole at $s = 1$ with residue independent of A. The pole may be eliminated by considering the function $Z_A(s)$, defined as follows.

Definition 1.14 (Differenced ray class zeta function)**.** Let R be the element of $\mathrm{Cl}_{\mathfrak{c}S}(\mathcal{O}_K)$ defined by

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}. \tag{1.37}$$

Define the *differenced zeta function of* A to be

$$Z_A(s) = \zeta(s, A) - \zeta(s, RA). \tag{1.38}$$

The function $Z_A(s)$ extends to a holomorphic function on the whole complex plane. The rank 1 abelian Stark conjecture says that $Z_A'(0)$ is the logarithm of an algebraic unit.

Conjecture 1.15 (Stark [12])**.** *Let K be a real quadratic field and $\{\rho_1, \rho_2\}$ the real embeddings of K. If R is not the identity of $\mathrm{Cl}_{\mathfrak{c}\infty_2}(\mathcal{O}_K)$, then $Z_A'(0) = \log(\rho_1(\varepsilon_A))$ for an algebraic unit ε_A generating the ray class field $L_{\mathfrak{c}\infty_2}$ corresponding to $\mathrm{Cl}_{\mathfrak{c}\infty_2}(\mathcal{O}_K)$. The units are compatible with the Artin map: $\varepsilon_{\mathrm{id}}^{\mathrm{Art}(A)} = \varepsilon_A$.*

The specialisation of the indefinite zeta function to a differenced real quadratic zeta function is given by the following result, which is Theorem 1.3 of [5].

Theorem 1.16. *For each $A \in \mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}$ and integral ideal $\mathfrak{b} \in A^{-1}$, there exists a real symmetric 2×2 matrix M, vectors $c_1, c_2 \in \mathbb{R}^2$, and $q \in \mathbb{Q}^2$ such that*

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s). \tag{1.39}$$

We may use Theorem 1.16 to compute presumptive Stark units $\exp(Z_A'(0))$. Specifically,

Corollary 1.17. *Under the specialisation given by Theorem 1.16,*

$$Z_A'(0) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, 0). \tag{1.40}$$

Proof. Take the limit of eq. (1.39) as $s \to 0$. □

We give an example of such a computation in section 4.

1.6. **Kronecker limit formulas for indefinite zeta functions.** The Kronecker limit formula for indefinite zeta functions is more complicated than the definite case. It involves the dilogarithm function and a rapidly convergent integral of a logarithm of an infinite product. We also require the following definition of the function $\kappa_\Omega^c(v)$, which is the square root of a rational function and will appear as a factor in the integrand.

Definition 1.18. Suppose $\Omega = M + iN \in \mathcal{H}_2^{(1)}$, $c \in \mathbb{C}^2$ satisfying $Q_M(c) < 0$, $v \in \mathbb{C}^2$, and $s \in \mathbb{C}$. Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} M c c^\top M$. Then, we define

$$\kappa_\Omega^c(v) = \frac{c^\top M v}{4\pi i \sqrt{-Q_M(c)} Q_\Omega(v) \sqrt{-2i Q_{\Lambda_c}(v)}}. \tag{1.41}$$

We now state the formula.

Theorem 1.19 (Indefinite Kronecker limit formula)**.** *Let* $\Omega = N + iM =\in \mathcal{H}_2^{(1)}$, $p = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, *and* $c_1, c_2 \in \mathbb{C}^2$ *such that* $\overline{c_j}^\top \operatorname{Im}\Omega c_j < 0$. *For* $c = c_1, c_2$, *factor the quadratic form*

$$Q_{\Lambda_c}\begin{pmatrix} \xi \\ 1 \end{pmatrix} = \alpha(c)(\xi - \tau_1(c))(\xi - \tau_2(c)), \tag{1.42}$$

where $\tau^+(c)$ *is in the upper half-plane and* $\tau^-(c)$ *is in the lower half-plane. Then,*

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = I^+(c_2) - I^-(c_2) - I^+(c_1) + I^-(c_1), \tag{1.43}$$

where

$$I^\pm(c) = -\operatorname{Li}_2(e(\pm p_1))\kappa_\Omega^c\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.44}$$

$$+ 2i \int_0^\infty \left(\operatorname{Log}\varphi_{p_1,\pm p_2}\right)(\pm\tau^\pm(c) + it)\kappa_\Omega^c\begin{pmatrix} \pm(\tau^\pm(c) + it) \\ 1 \end{pmatrix} dt. \tag{1.45}$$

The function $\varphi_{p_1,p_2} : \mathcal{H} \to \mathbb{C}$ *is defined by the a product expansion,*

$$\varphi_{p_1,p_2}(\xi) := (1 - e(p_1\xi_t + p_2)) \prod_{d=1}^\infty \frac{1 - e\left((d + p_1)\xi + p_2\right)}{1 - e\left((d - p_1)\xi - p_2\right)}, \tag{1.46}$$

and its logarithm $\left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi)$ *is the unique continuous branch with the property*

$$\lim_{\xi \to i\infty} \left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\xi) = \begin{cases} \log(1 - e(p_2)) & \text{if } p_1 = 0, \\ 0 & \text{if } p_1 \neq 0. \end{cases} \tag{1.47}$$

Here $\log(1 - e(p_2))$ *is the standard principal branch.*

The following specialisation looks somewhat simpler and contains all of the cases of arithmetic zeta functions $Z_A(s)$ associated to real quadratic fields.

Theorem 1.20 (Indefinite Kronecker limit formula, pure imaginary case)**.** *Let M be a 2×2 real matrix of signature $(1,1)$, and let $\Omega = iM$. Let $p = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \in \mathbb{R}^2$, and $c_1, c_2 \in \mathbb{R}^2$ such that $c_j^\top M c_j < 0$. Then,*

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = 2i \operatorname{Im}\left(I(c_2) - I(c_1)\right), \tag{1.48}$$

where

$$I(c) = -\operatorname{Li}_2(e(p_1))\kappa_\Omega^c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.49}$$

$$+ 2i \int_0^\infty \left(\operatorname{Log}\varphi_{p_1,p_2}\right)(\tau(c) + it)\kappa_\Omega^c \begin{pmatrix} \tau(c) + it \\ 1 \end{pmatrix} dt. \tag{1.50}$$

Here, $\operatorname{Log}\varphi_{p_1,p_2}$ and κ_Ω^c are defined as in the statement of Theorem 1.19, and $\xi = \tau(c)$ is the unique root of the quadratic polynomial $Q_{\Lambda_c}\left(\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right)$ in the upper half plane.

As in the definite case, it is straightforward to use the functional equation for the indefinite zeta function to rephrase Theorem 1.19 as a formula for $\hat{\zeta}_{0,q}^{c_1,c_2}(\Omega, 0)$. We illustrate the computation of indefinite zeta values at $s = 0$ by example in section 4.

2. Kronecker limit formulas for definite zeta functions

In this section, we prove the Kronecker limit formulas for definite zeta functions at $s = 1$ and $s = 0$, from which the classical second Kronecker limit formula follows. The method of proof at $s = 1$ is to compute the Fourier series in a single real variable ξ for a definite theta function with respect to an action by a one-parameter unipotent subgroup $\{T^\xi\}$ of $\mathbf{SL}_2(\mathbb{R})$. We then take the Mellin transform term-by-term and send $s \to 1$. The formula at $s = 0$ follows from the formula at $s = 1$ by the functional equation for the definite zeta function, which we also prove.

2.1. Fourier series of a unipotent transform of a definite theta function.

Consider the definite (Riemann) theta function in dimension $g = 2$, for $z \in \mathbb{C}^2$ and $\Omega \in \mathcal{H}_2^{(0)}$;

$$\Theta(z, \Omega) = \sum_{n \in \mathbb{Z}^2} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right). \tag{2.1}$$

Let $T^\xi = \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right)$ for $\xi \in \mathbb{R}$, and fix z and Ω. In this section, we will calculate the Fourier expansion of the function

$$g(\xi) = \Theta(\left(T^\xi\right)^\top z; \left(T^\xi\right)^\top \Omega T^\xi) = \sum_{n \in \mathbb{Z}^2} e\left(\frac{1}{2}\left(T^\xi n\right)^\top \Omega \left(T^\xi n\right) + \left(T^\xi n\right)^\top z\right). \tag{2.2}$$

We have $g(\xi + 1) = g(\xi)$ by an integral change of basis on $\mathbb{Z}^2$. We will compute the Fourier coefficients

$$a_k = \int_0^1 g(\xi) e(-k\xi)\, d\xi. \tag{2.3}$$

Proposition 2.1. *The Fourier coefficeints of $g(\xi)$ are given by the following formulas. If $k \neq 0$,*

$$a_k = \frac{e\left(\frac{2k\omega_{12} - z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}} \sum_{\substack{d \in \mathbb{Z} \\ d \mid k}} e\left(\frac{1}{2\omega_{11}}\left((\det \Omega) d^2 + 2(\omega_{11} z_2 - \omega_{12} z_1) d + 2 z_1 \frac{k}{d} - \frac{k^2}{d^2}\right)\right). \tag{2.4}$$

For $k = 0$, and using $\vartheta(z, \omega)$ to denote the Jacobi theta function, we have

$$a_0 = \vartheta(z_1, \omega_{11}) + \frac{e\left(\frac{-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}}\left(\vartheta\left(\frac{\omega_{11} z_2 - \omega_{12} z_1}{\omega_{11}}, \frac{\det \Omega}{\omega_{11}}\right) - 1\right). \tag{2.5}$$

Proof. Express $\Omega = \left(\begin{smallmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{smallmatrix}\right)$, $n = \left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right)$, $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$. Write $g(\xi) = \sum_{n_2=-\infty}^{\infty} g_{n_2}(\xi)$, where $g_j(\xi)$ is the sum over the terms with $n_2 = j$. First, calculate $g_0(\xi)$:

$$g_0(\xi) = \sum_{n_1=-\infty}^{\infty} e\left(\frac{1}{2}\omega_{11} n_1^2 + n_1 z_1\right) \tag{2.6}$$

$$= \vartheta(z_1, \omega_{11}), \tag{2.7}$$

where ϑ is the Jacobi theta function. We may write $g_{n_2}(\xi) e(-k\xi)$ as

$$\begin{aligned} &g_{n_2}(\xi) e(-k\xi) \\ &= \sum_{n_1=-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}(n_1 + n_2\xi)^2 + \omega_{12} n_2 (n_1 + n_2 \xi)\right. \\ &\qquad\qquad \left. + \frac{\omega_{22}}{2} n_2^2 + (n_1 + n_2\xi) z_1 + n_2 z_2 - k\xi\right) &(2.8) \\ &= \sum_{n_1=-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}(n_1 + n_2\xi)^2 + \left(\omega_{12} n_2 + z_1 - \frac{k}{n_2}\right)(n_1 + n_2\xi)\right. \\ &\qquad\qquad \left. + \left(\frac{\omega_{22}}{2} n_2^2 + n_2 z_2 + \frac{k n_1}{n_2}\right)\right) &(2.9) \\ &= \sum_{n_1=-\infty}^{\infty} b_{n_1,n_2} e\left(\frac{\omega_{11}}{2}\left((n_1 + n_2\xi) + \frac{\omega_{12} n_2 + z_1 - k/n_2}{\omega_{11}}\right)^2\right), &(2.10) \end{aligned}$$

where $b_{n_1,n_2} = e\left(\left(\frac{\omega_{22}}{2} n_2^2 + n_2 z_2 + \frac{k n_1}{n_2}\right) - \frac{(\omega_{12} n_2 + z_1 - k/n_2)^2}{2\omega_{11}}\right)$. Note that $b_{n_1+n_2,n_2} = b_{n_1,n_2}$, and, moreover,

$$g_{n_2}(\xi) e(-k\xi) = \sum_{n_1=0}^{n_2-1} b_{n_1,n_2} \sum_{j=-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}\left((n_1 + n_2(\xi + j)) + \frac{\omega_{12} n_2 + z_1 - k/n_2}{\omega_{11}}\right)^2\right). \tag{2.11}$$

Thus,

$$\begin{aligned}
&\int_0^1 g_{n_2}(\xi)e(-k\xi)\,d\xi \\
&= \sum_{n_1=0}^{n_2-1} b_{n_1,n_2} \int_{-\infty}^{\infty} e\left(\frac{\omega_{11}}{2}\left((n_1+n_2\xi)+\frac{\omega_{12}n_2+z_1-k/n_2}{\omega_{11}}\right)^2\right)d\xi && (2.12)\\
&= \sum_{n_1=0}^{n_2-1} \frac{b_{n_1,n_2}}{\sqrt{-i\omega_{11}n_2^2}}, && (2.13)
\end{aligned}$$

by Corollary 2.6 of [5].

$$\begin{aligned}
\int_0^1 g_{n_2}(\xi)e(-k\xi)\,d\xi &= \frac{1}{\sqrt{-i\omega_{11}}\,|n_2|}\sum_{n_1=0}^{n_2-1} b_{n_1,n_2} && (2.14)\\
&= \frac{e\left(\left(\frac{\omega_{22}}{2}n_2^2+n_2z_2\right)-\frac{(\omega_{12}n_2+z_1-k/n_2)^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}\,|n_2|}\sum_{n_1=0}^{n_2-1} e\left(\frac{kn_1}{n_2}\right) && (2.15)\\
&= \begin{cases} \frac{1}{\sqrt{-i\omega_{11}}}e\left(\left(\frac{\omega_{22}}{2}n_2^2+n_2z_2\right)-\frac{(\omega_{12}n_2+z_1-k/n_2)^2}{2\omega_{11}}\right), & \text{if } n_2|k;\\ 0, & \text{else.}\end{cases} && (2.16)
\end{aligned}$$

Thus, for $k \neq 0$, we have $\int_0^1 \vartheta(z_1,\omega_{11})e(-k\xi)\,d\xi = 0$ and

$$\begin{aligned}
a_k &= \sum_{\substack{d\in\mathbb{Z}\\ d|k}} \frac{1}{\sqrt{-i\omega_{11}}}e\left(\left(\frac{\omega_{22}}{2}d^2+dz_2\right)-\frac{(\omega_{12}d+z_1-k/d)^2}{2\omega_{11}}\right) && (2.17)\\
&= \frac{e\left(\frac{2k\omega_{12}-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}}\sum_{\substack{d\in\mathbb{Z}\\ d|k}} e\left(\frac{1}{2\omega_{11}}\left((\det\Omega)d^2+2(\omega_{11}z_2-\omega_{12}z_1)d\right.\right.\\
&\qquad\left.\left.+2z_1\frac{k}{d}-\frac{k^2}{d^2}\right)\right). && (2.18)
\end{aligned}$$

For $k=0$, we have $\int_0^1 \vartheta(z_1,\omega_{11})e(-k\xi)\,d\xi = \vartheta(z_1,\omega_{11})$ and

$$\begin{aligned}
a_0 &= \vartheta(z_1,\omega_{11}) + \sum_{d\in\mathbb{Z}\setminus\{0\}} \frac{1}{\sqrt{-i\omega_{11}}}e\left(\left(\frac{\omega_{22}}{2}d^2+dz_2\right)-\frac{(\omega_{12}d+z_1)^2}{2\omega_{11}}\right) && (2.19)\\
&= \vartheta(z_1,\omega_{11}) + \frac{e\left(\frac{-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}}\sum_{d\in\mathbb{Z}\setminus\{0\}} e\left(\frac{\det\Omega}{\omega_{11}}d^2+\frac{\omega_{11}z_2-\omega_{12}z_1}{\omega_{11}}d\right) && (2.20)\\
&= \vartheta(z_1,\omega_{11}) + \frac{e\left(\frac{-z_1^2}{2\omega_{11}}\right)}{\sqrt{-i\omega_{11}}}\left(\vartheta\left(\frac{\omega_{11}z_2-\omega_{12}z_1}{\omega_{11}},\frac{\det\Omega}{\omega_{11}}\right)-1\right). && (2.21)
\end{aligned}$$

This completes the proof of the proposition. □

From now on, we will use the notation $\sum_{d|k}$ in place of $\sum_{\substack{d\in\mathbb{Z}\\ d|k}}$. This is nonstandard—we are summing over all integral divisors of k, not just positive divisors. A sum over the divisors of 0 is a sum over all integers.

Use the definite theta with characteristics to define a function of $\xi, t \in \mathbb{R}$,

$$h(\xi,t) := \Theta_{(T^\xi)^\top p, T^{-\xi}q}(t(T^\xi)^\top \Omega T^\xi) \tag{2.22}$$

$$= e\left(\frac{t}{2}q^\top \Omega q + p^\top q\right)\Theta\left((T^\xi)^\top(p+t\Omega q), t(T^\xi)^\top \Omega T^\xi\right). \tag{2.23}$$

Write this function as a Fourier series,

$$h(\xi,t) = \sum_{k=-\infty}^{\infty} b_k(t)e(k\xi). \tag{2.24}$$

The Fourier coefficients of $h(\xi,t)$ are given by the following corollary.

Corollary 2.2. *If $k \neq 0$, then*

$$b_k(t) = \frac{t^{-1/2}}{\sqrt{-i\omega_{11}}}\sum_{d|k} e\left(\frac{(\det\Omega)(q_2+d)^2}{2\omega_{11}}t + \frac{(\omega_{11}p_2 - \omega_{12}(p_1-k/d))(q_2+d)+\omega_{11}q_1k/d}{\omega_{11}} - \frac{(p_1-k/d)^2}{2\omega_{11}}t^{-1}\right). \tag{2.25}$$

For $k=0$, we have

$$b_0(t) = \sum_{n=-\infty}^{\infty} e\left(\left(\frac{\omega_{11}}{2}(q_1+n)^2 + \omega_{12}(q_1+n)q_2 + \frac{\omega_{22}}{2}q_2^2\right)t + (p_1q_1+p_2q_2+p_1n)\right) \tag{2.26}$$

$$+ \frac{t^{-1/2}}{\sqrt{-i\omega_{11}}}\sum_{d\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\det\Omega)(q_2+d)^2}{2\omega_{11}}t + \frac{(\omega_{11}p_2-\omega_{12}p_1)(q_2+d)}{\omega_{11}} - \frac{p_1^2}{2\omega_{11}}t^{-1}\right).$$

Proof. Follows from Proposition 2.1. □

2.2. **Taking the Mellin transform term-by-term.** Next, we will shift our focus from theta functions to zeta functions. We will need to take a Mellin transform term-by-term in an infinite sum, and, to do this, we will need an absolute convergence result. First, we need the following technical inequality.

Lemma 2.3. *Let $\Omega = \left(\begin{smallmatrix}\omega_{11} & \omega_{12}\\ \omega_{12} & \omega_{22}\end{smallmatrix}\right) \in \mathcal{H}_2^{(0)}$. Then*

$$\mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)\mathrm{Im}\left(\frac{\det\Omega}{\omega_{11}}\right) > \left(\mathrm{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)^2. \tag{2.27}$$

Proof. Express Ω in terms of its real and imaginary parts,

$$\begin{pmatrix}\omega_{11} & \omega_{12}\\ \omega_{12} & \omega_{22}\end{pmatrix} = \begin{pmatrix}n_{11} & n_{12}\\ n_{12} & n_{22}\end{pmatrix} + i\begin{pmatrix}m_{11} & m_{12}\\ m_{12} & m_{22}\end{pmatrix}. \tag{2.28}$$

Note that $m_{11} \neq 0$ because $m_{11}m_{22} - m_{12}^2 = \det M > 0$, and thus $\omega_{11} \neq 0$. By an algebraic calculation,

$$\operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det\Omega}{\omega_{11}}\right) - \left(\operatorname{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)^2 = \frac{m_{11}m_{22} - m_{12}^2}{n_{11}^2 + m_{11}^2}. \tag{2.29}$$

Now, $m_{11}m_{22} - m_{12}^2 = \det M$ is positive, and so is $n_{11}^2 + m_{11}^2$. Thus, the inequality eq. (2.27) holds. □

Here is another inequality that we will need later.

Lemma 2.4. *Let* $\Omega = \left(\begin{smallmatrix}\omega_{11} & \omega_{12}\\ \omega_{12} & \omega_{22}\end{smallmatrix}\right) \in \mathcal{H}_2^{(0)}$. *The two roots of* $Q_\Omega\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right) = 0$ *are* $\tau_1 = \frac{-\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}$ *and* $\tau_2 = \frac{-\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}$. *Then,* $\operatorname{Im}(\tau_1) > 0 > \operatorname{Im}(\tau_2)$.

Proof. We have $Q_\Omega\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right) = \omega_{11}z^2 + 2\omega_{12}z + \omega_{22}$, and the expressions for the roots come from the quadratic formula.

For any complex numbers $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$, $(\operatorname{Im}(\alpha\beta))^2 - \operatorname{Im}(\alpha^2)\operatorname{Im}(\beta^2) = (a_1b_2 - a_2b_1)^2 \geq 0$. Thus, $(\operatorname{Im}(\alpha\beta))^2 \geq \operatorname{Im}(\alpha^2)\operatorname{Im}(\beta^2)$.

In particular, taking $\alpha = \frac{1}{\sqrt{-\omega_{11}}}$ and $\beta = \frac{\sqrt{\det(-i\Omega)}}{\sqrt{-\omega_{11}}}$ (for any choice of $\sqrt{-\omega_{11}}$), we obtain the inequality

$$\left(\operatorname{Im}\left(\frac{\sqrt{\det(-i\Omega)}}{\omega_{11}}\right)\right)^2 \geq \operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det(-i\Omega)}{-\omega_{11}}\right) \tag{2.30}$$

$$= \operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det(\Omega)}{\omega_{11}}\right). \tag{2.31}$$

Appealing to Lemma 2.3, we see by transitivity that

$$\left(\operatorname{Im}\left(\frac{\sqrt{\det(-i\Omega)}}{\omega_{11}}\right)\right)^2 > \left(\operatorname{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)^2. \tag{2.32}$$

By subtracting the left-hand side and factoring, this inequality may be rewritten as $0 > \operatorname{Im}(\tau_1)\operatorname{Im}(\tau_2)$. So $\operatorname{Im}(\tau_1)$ and $\operatorname{Im}(\tau_2)$ are always nonzero real numbers with opposite signs. In the special case $\Omega = \left(\begin{smallmatrix}i & 0\\ 0 & i\end{smallmatrix}\right)$, $\tau_1 = i$ and $\tau_2 = -i$. Since $\mathcal{H}_2^{(0)}$ is connected, we always have $\operatorname{Im}(\tau_1) > 0 > \operatorname{Im}(\tau_2)$. □

We will encounter Bessel functions in both our absolute convergence argument and our calculation of the Fourier coefficients in the next section. Let K_s denote the K-Bessel function. That is, for $\operatorname{Re}(\alpha) > 0$,

$$K_s(\alpha) := \frac{1}{2}\int_0^\infty \exp\left(-\frac{\alpha}{2}\left(t + t^{-1}\right)\right) t^s \,\frac{dt}{t}. \tag{2.33}$$

This function satisfies the identities $K_s(\alpha) = K_{-s}(\alpha)$ and $K_{\frac{1}{2}}(\alpha) = \sqrt{\frac{\pi}{2\alpha}}e^{-\alpha}$. It also has exponential decay in α; specifically, $|K_s(\alpha)| = o(\exp(-\alpha))$ as $\operatorname{Re}(\alpha) \to \infty$. (See p. 66 of [2] and chapter 4 of [1].)

We can use the Bessel function to write certain integrals in a more compact form.

Lemma 2.5. *Suppose $a, b \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$. Taking the standard branch of all power functions with a branch cut along the negative real axis,*

$$\int_0^\infty \exp\left(-\left(at + bt^{-1}\right)\right) t^s \,\frac{dt}{t} = 2(b/a)^{s/2} K_s(2\sqrt{ab}). \tag{2.34}$$

Proof. Substitute $t = \sqrt{\frac{b}{a}}u$.

$$\begin{aligned}
&\int_0^\infty \exp\left(-\left(at + bt^{-1}\right)\right) t^s \,\frac{dt}{t} \\
&= (b/a)^{s/2} \lim_{N\to\infty} \int_0^{\sqrt{\frac{b}{a}}N} \exp\left(-\sqrt{ab}\left(u+u^{-1}\right)\right) u^s \,\frac{du}{u} &(2.35)\\
&= (b/a)^{s/2} \lim_{N\to\infty} \left(\int_0^{\sqrt{\left|\frac{b}{a}\right|}N} + \int_{\sqrt{\left|\frac{b}{a}\right|}N}^{\sqrt{\frac{b}{a}}N}\right) \exp\left(-\sqrt{ab}\left(u+u^{-1}\right)\right) u^s \,\frac{du}{u} &(2.36)\\
&= (b/a)^{s/2} \lim_{N\to\infty} \int_0^{\sqrt{\left|\frac{b}{a}\right|}N} \exp\left(-\sqrt{ab}\left(u+u^{-1}\right)\right) u^s \,\frac{du}{u} &(2.37)\\
&= 2(b/a)^{s/2} K_s(2\sqrt{ab}). &(2.38)
\end{aligned}$$

In eq. (2.36), we used the bound

$$\left|\int_{\sqrt{\left|\frac{b}{a}\right|}N}^{\sqrt{\frac{b}{a}}N} \exp\left(-\sqrt{ab}\left(u+u^{-1}\right)\right) u^s \,\frac{du}{u}\right| \le \exp(-bN)\mathrm{poly}(N) \to 0 \tag{2.39}$$

as $N \to \infty$, where $\mathrm{poly}(N)$ denotes some polynomial. ☐

Now, we'll prove an absolute convergence result that will allow us to reverse the order of summation/integration.

Proposition 2.6. *For any $\sigma \in \mathbb{R}$,*

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} < \infty. \tag{2.40}$$

Proof. We bound $b_k(t)$ by

$$\begin{aligned}
|b_k(t)| \le |\omega_{11}|^{-\frac{1}{2}}\, t^{-\frac{1}{2}} \sum_{d|k} \exp\Bigg(-\pi\Bigg(\mathrm{Im}\left(\frac{\det\Omega}{\omega_{11}}\right)(d+q_2)^2 t &&(2.41)\\
+2\,\mathrm{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)(k/d-p_1)(d+q_2) + \mathrm{Im}\left(\frac{-1}{\omega_{11}}\right)(k/d-p_1)^2 t^{-1}\Bigg)\Bigg). &&(2.42)
\end{aligned}$$

Thus, we have (for some polynomial function $p(k)$)

$$\begin{aligned}\int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} &\le p(k) \exp\left(2\pi \operatorname{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)(k/d - p_1)(d+p_2)\right)\\ &\cdot K_\sigma\left(2\pi\sqrt{\operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det\Omega}{\omega_{11}}\right)}\,|(k/d-p_1)(d+p_2)|\right) \qquad (2.43)\\ &\le p(k)\exp\left(-2\pi\left(\sqrt{\operatorname{Im}\left(\frac{-1}{\omega_{11}}\right)\operatorname{Im}\left(\frac{\det\Omega}{\omega_{11}}\right)} \pm \operatorname{Im}\left(\frac{\omega_{12}}{\omega_{11}}\right)\right)\right.\\ &\qquad \left. |(k/d-p_1)(d+p_2)|\right). \qquad (2.44)\end{aligned}$$

In the second line, we used the fact that, as $\alpha \to \infty$, $K_\sigma(\alpha) = o(\exp(-\alpha))$. Now, by Lemma 2.3, there is a constant $\varepsilon > 0$ so that

$$\int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} \le \exp\left(\varepsilon\, |(k/d-p_1)(d+p_2)|\right). \qquad (2.45)$$

Thus,

$$\sum_{k=-\infty}^{\infty} \int_0^\infty |b_k(t)|\, t^\sigma \frac{dt}{t} \le \sum_{d_1\neq 0}\sum_{d_2 \neq 0} \exp\left(\varepsilon\, |(d_1-p_1)(d_2+p_2)|\right) < \infty. \qquad (2.46)$$

This completes the proof of the proposition. □

Now we may compute the Fourier series in ξ for $\zeta_{(T^\xi)^\top p, T^{-\xi} q}\left((T^\xi)^\top \Omega T^\xi, s\right)$.

Proposition 2.7. *The Fourier coefficients $\beta_k(s)$ of $\zeta_{(T^\xi)^\top p, T^{-\xi} q}\left((T^\xi)^\top \Omega T^\xi, s\right)$ are given by the following formulas. If $k \neq 0$, then*

$$\begin{aligned}\beta_k(s) = \frac{2}{\sqrt{-i\omega_{11}}} \sum_{d|k} e&\left(\frac{(\omega_{11}p_2 - \omega_{12}(p_1 - k/d))(q_2+d) + \omega_{11} q_1 k/d}{\omega_{11}}\right)\\ &\cdot (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \left|\frac{p_1 - k/d}{q_2+d}\right|^{s-\frac{1}{2}}\\ &\cdot K_{s-\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-k/d)(q_2+d)|\right). \qquad (2.47)\end{aligned}$$

For $k = 0$,

$$\int_0^\infty b_0(t) t^s \frac{dt}{t}$$

$$= (2\pi)^{-s}\Gamma(s) \sum_{n=-\infty}^{\infty} e(p_1q_1 + p_2q_2 + p_1n) Q_\Omega \begin{pmatrix} q_1 + n \\ q_2 \end{pmatrix}^{-s} \tag{2.48}$$

$$+ \frac{2}{\sqrt{-i\omega_{11}}} \sum_{d \in \mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2 - \omega_{12}p_1)(q_2 + d)}{\omega_{11}}\right) (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \tag{2.49}$$

$$\cdot \left|\frac{p_1}{q_2 + d}\right|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|p_1(q_2 + d)|\right). \tag{2.50}$$

Proof. It follows from Proposition 2.6 that we may take the Mellin transform term-by-term.

$$\zeta_{(T^\xi)^\top p, T^{-\xi} q}\left((T^\xi)^\top \Omega T^\xi, s\right) = \int_0^\infty h(\xi, t) t^s \frac{dt}{t} = \sum_{k=-\infty}^{\infty} \left(\int_0^\infty b_k(t) t^s \frac{dt}{t}\right) e(k\xi). \tag{2.51}$$

The formulas follow by Lemma 2.5. □

2.3. **Proof of the Kronecker limit formula at $s = 1$.** We will need a standard result on the values of the polylogarithm $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} k^{-s} z^k$ at positive integers $s = n$.

Proposition 2.8. *Suppose $n \in \mathbb{Z}$, $n \geq 1$, $x \in \mathbb{R}$, and $0 \leq x \leq 1$. Then,*

$$\mathrm{Li}_n(e(x)) + (-1)^n \mathrm{Li}_n(e(-x)) = -\frac{(2\pi i)^n}{n!} B_n(x), \tag{2.52}$$

where $B_{2n}(x)$ is the $(2n)$th Bernoulli polynomial.

Proof. A proof may be found in [1]. □

We will only need this result at $s = 2$.

Corollary 2.9. *If $x \in \mathbb{R}$, and $\{x\}$ denotes the fractional part of x, then*

$$\mathrm{Li}_2(e(x)) + \mathrm{Li}_2(e(-x)) = 2\pi^2 \left(\{x\}^2 - \{x\} + \frac{1}{6}\right). \tag{2.53}$$

Proof. Plug $n = 1$ into Proposition 2.8. □

Using the change of variables $(d_1, d_2) = (\frac{n}{d}, d)$, we have

$$\begin{aligned}
&\hat{\zeta}_{p,q}(\Omega, s) \\
&= (2\pi)^{-s}\Gamma(s) \sum_{n=-\infty}^{\infty} e(p_1 q_1 + p_2 q_2 + p_1 n) Q_{-i\Omega}\begin{pmatrix} q_1 + n \\ q_2 \end{pmatrix}^{-s} \\
&\quad + \frac{2}{\sqrt{-i\omega_{11}}} \sum_{k\in\mathbb{Z}} \sum_{\substack{d|k \\ d\neq 0}} e\left(\frac{(\omega_{11}p_2 - \omega_{12}(p_1 - k/d))(q_2 + d) + \omega_{11}q_1 k/d}{\omega_{11}}\right) \\
&\qquad \cdot (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \left|\frac{p_1 - k/d}{q_2 + d}\right|^{s-\frac{1}{2}} \\
&\qquad \cdot K_{s-\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1 - k/d)(q_2 + d)|\right) \qquad (2.54) \\
&= (2\pi)^{-s}\Gamma(s) \sum_{n=-\infty}^{\infty} e(p_1 q_1 + p_2 q_2 + p_1 n) Q_{-i\Omega}\begin{pmatrix} q_1 + n \\ q_2 \end{pmatrix}^{-s} \\
&\quad + \frac{2}{\sqrt{-i\omega_{11}}} \sum_{d_1\in\mathbb{Z}} \sum_{d_2\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2 - \omega_{12}(p_1 - d_1))(q_2 + d_2) + \omega_{11}q_1 d_1}{\omega_{11}}\right) \\
&\qquad \cdot (\det(-i\Omega))^{-\frac{s}{2}+\frac{1}{4}} \left|\frac{p_1 - d_1}{q_2 + d_2}\right|^{s-\frac{1}{2}} \\
&\qquad \cdot K_{s-\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1 - d_1)(q_2 + d_2)|\right). \qquad (2.55)
\end{aligned}$$

Specialise to the case when $s=1$, $q=\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$, and $p_1p_2\neq 0$. Now,

$$\begin{aligned}
\hat{\zeta}_{p,0}(\Omega,1) &= (2\pi)^{-1}\Gamma(1)\sum_{n\in\mathbb{Z}\setminus\{0\}} e(p_1 n)\left(\frac{-i\omega_{11}}{2}n^2\right)^{-1} \\
&\quad + \frac{2}{\sqrt{-i\omega_{11}}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right)(\det(-i\Omega))^{-\frac{1}{4}} \\
&\qquad \cdot\left|\frac{p_1-d_1}{d_2}\right|^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{2\pi i}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right) && (2.56) \\
&= \frac{1}{-\pi i\omega_{11}}\left(\mathrm{Li}_2(e(p_1))-\mathrm{Li}_2(e(-p_1))\right) \\
&\quad + \frac{2}{\sqrt{-i\omega_{11}}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}} e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right)(\det(-i\Omega))^{-\frac{1}{4}} \\
&\qquad \cdot\left|\frac{p_1-d_1}{d_2}\right|^{\frac{1}{2}}\left(\frac{\sqrt{-i\omega_{11}}}{2}\det(-i\Omega)^{-1/4}\,|(p_1-d_1)d_2|^{-1/2}\right) \\
&\qquad \cdot e\left(\frac{-1}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right) && (2.57) \\
&= \frac{1}{-\pi i\omega_{11}}2\pi^2\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right) \\
&\quad + \frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}}\frac{1}{|d_2|}e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right. \\
&\qquad \left.-\frac{1}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right) && (2.58) \\
&= \frac{2\pi}{-i\omega_{11}}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right) \\
&\quad + \frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1\in\mathbb{Z}}\sum_{d_2\in\mathbb{Z}\setminus\{0\}}\frac{1}{|d_2|}e\left(\frac{(\omega_{11}p_2-\omega_{12}(p_1-d_1))d_2}{\omega_{11}}\right. \\
&\qquad \left.-\frac{1}{\omega_{11}}\sqrt{\det(-i\Omega)}\,|(p_1-d_1)d_2|\right). && (2.59)
\end{aligned}$$

Split the series up into four pieces.

$$\begin{aligned}
\hat{\zeta}_{p,0}(\Omega,1) &= \frac{2\pi}{-i\omega_{11}}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&\quad+\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\sum_{d_2>0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2}\\
&\quad+\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\sum_{d_2>0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2}\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\sum_{d_2<0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2}\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\sum_{d_2<0}\frac{1}{d_2}e\left(p_2+\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)^{d_2} \qquad (2.60)\\
&= \frac{2\pi}{-i\omega_{11}}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\log\left(1-e\left(p_2+\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\log\left(1-e\left(p_2+\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1>p_1}\log\left(1-e\left(-p_2-\frac{\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right)\\
&\quad-\frac{1}{\sqrt{\det(-i\Omega)}}\sum_{d_1<p_1}\log\left(1-e\left(-p_2-\frac{\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}(d_1-p_1)\right)\right) \qquad (2.61)
\end{aligned}$$

Let $\tau_1=\frac{-\omega_{12}-\sqrt{\det(-i\Omega)}}{\omega_{11}}$ and $\tau_2=\frac{-\omega_{12}+\sqrt{\det(-i\Omega)}}{\omega_{11}}$, so that $Q_\Omega\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right)=\frac{\omega_{11}}{2}(z-\tau_1)(z-\tau_2)$. Then $\frac{\tau_1-\tau_2}{2i}=\frac{\sqrt{\det(-i\Omega)}}{-i\omega_{11}}$, and

$$\begin{aligned}
-\sqrt{\det(-i\Omega)}\hat{\zeta}_{p,0}(\Omega,1) &= -2\pi\frac{\tau_1-\tau_2}{2i}\left(\{p_1\}^2-\{p_1\}+\frac{1}{6}\right) \qquad (2.62)\\
&+\sum_{d_1>p_1}\log\left(1-e\left(p_2-\tau_2(d_1-p_1)\right)\right)+\sum_{d_1<p_1}\log\left(1-e\left(p_2-\tau_1(d_1-p_1)\right)\right)\\
&+\sum_{d_1>p_1}\log\left(1-e\left(-p_2+\tau_1(d_1-p_1)\right)\right)+\sum_{d_1<p_1}\log\left(1-e\left(-p_2+\tau_2(d_1-p_1)\right)\right).
\end{aligned}$$

Assume that $0 \leq p_1 < 1$. The first term may be rewritten as

$$-2\pi\frac{\tau_1 - \tau_2}{2i}\left(\{p_1\}^2 - \{p_1\} + \frac{1}{6}\right) = \log\left(e(-p_2/2)e(\tau_1)^{p_1^2/2+1/12}e(p_2 - p_1\tau_1)^{1/2}\right) + \log\left(e(-p_2/2)e(-\tau_2)^{p_1^2/2+1/12}e(p_2 + p_1\tau_2)^{1/2}\right) \quad (2.63)$$

So, the whole thing can be written

$$-\sqrt{\det(-i\Omega)}\hat{\zeta}_{p,0}(\Omega, 1) = \left(\operatorname{Log} f_{p_1,p_2}\right)(\tau_1) + \left(\operatorname{Log} f_{p_1,p_2}\right)(-\tau_2), \quad (2.64)$$

where

$$f_{p_1,p_2}(\tau) = e(-p_2/2)e(\tau)^{p_1^2/2+1/12}\left(e(p_2 - p_1\tau)^{1/2} - e(p_2 - p_1\tau)^{-1/2}\right) \quad (2.65)$$

$$\cdot\prod_{d=1}^{\infty}\left(1 - e(\tau)^d e\left(p_2 - p_1\tau\right)\right)\left(1 - e(\tau)^d e\left(p_2 - p_1\tau\right)^{-1}\right), \quad (2.66)$$

and $(\operatorname{Log} f_{p_1,p_2})(\tau)$ is the unique holomorphic function on $\mathcal{H}$ such that $\exp\left((\operatorname{Log} f_{p_1,p_2})(\tau)\right) = f_{p_1,p_2}(\tau)$ and

$$(\operatorname{Log} f_p)(\tau) \sim \pi i\left(p_1^2 - p_1 + \frac{1}{6}\right)\tau \text{ as } \tau \to i\infty. \quad (2.67)$$

This proves the first part of Theorem 1.4.

Now rewrite $f_{p_1,p_2}(\tau)$ as a ϑ-function. The Jacobi triple product identity says,

Theorem 2.10. *For $z, w \in C$, $|z| < 1$, $w \neq 0$, the following identity holds:*

$$\prod_{d=1}^{\infty}\left(1 - z^{2d}\right)\left(1 - wz^{2d-1}\right)\left(1 - w^{-1}z^{2d-1}\right) = \sum_{n=-\infty}^{\infty}(-1)^n w^n z^{n^2}. \quad (2.68)$$

Proof. See Theorem 10.4.1 of [1]. □

Proposition 2.11. *If $0 \leq p_1, p_2 < 1$ and $\tau \in \mathcal{H}$, then*

$$f_{p_1,p_2}(\tau) = \frac{e\left(\left(p_1 - \frac{1}{2}\right)\left(p_2 + \frac{1}{2}\right)\right)\vartheta_{\frac{1}{2}+p_2,\frac{1}{2}-p_1}(\tau)}{\eta(\tau)}. \quad (2.69)$$

Proof. Let $u = e(\tau)$ and $v = e(p_2 - p_1\tau)$, and rewrite this formula as

$$f_{p_1,p_2}(\tau) = e\left(-\frac{p_2}{2}\right)u^{p_1^2/2+1/12}\left(v^{1/2} - v^{-1/2}\right)\prod_{d=1}^{\infty}\left(1 - u^d v\right)\left(1 - u^d v^{-1}\right). \quad (2.70)$$

Now, use the Jacobi triple product identity to rewrite the product as a sum.

$$\left(v^{1/2} - v^{-1/2}\right)\prod_{d=1}^{\infty}\left(1 - u^d v\right)\left(1 - u^d v^{-1}\right)$$

$$= \frac{v^{\frac{1}{2}}}{\prod_{d=1}^{\infty}(1 - u^d)}\prod_{d=1}^{\infty}(1 - (u^{\frac{1}{2}})^{2d})(1 - (u^{\frac{1}{2}})^{2d-1}(u^{\frac{1}{2}}v))(1 - (u^{\frac{1}{2}})^{2d-1}(u^{\frac{1}{2}}v)^{-1}) \quad (2.71)$$

$$= \frac{v^{1/2}u^{1/24}}{\eta(\tau)}\sum_{n=-\infty}^{\infty}(-1)^n u^{n^2/2+n/2}v^n, \quad (2.72)$$

using Theorem 2.10 in the last line. Thus,

$$f_{p_1,p_2}(\tau) = e\left(-\frac{p_2}{2}\right)\frac{v^{1/2}u^{p_1^2/2+1/8}}{\eta(\tau)}\sum_{n=-\infty}^{\infty} u^{n^2/2+n/2}v^n \tag{2.73}$$

$$= \frac{e\left(-\frac{p_2}{2}\right)}{\eta(\tau)}\sum_{n=-\infty}^{\infty} u^{n^2/2+n/2+p_1^2/2+1/8}v^{n+1/2}. \tag{2.74}$$

We have

$$\begin{aligned}
&(-1)^n u^{n^2/2+n/2+p_1^2/2+1/8}v^{n+1/2}\\
&= e\left(\left(\frac{n^2}{2}+\frac{n}{2}+\frac{p_1^2}{2}+\frac{1}{8}\right)\tau+\left(n+\frac{1}{2}\right)(p_2-p_1\tau)+\frac{n}{2}\right)\\
&= e\left(\left(n^2-2\left(p_1-\frac{1}{2}\right)n+p_1^2-p_1+\frac{1}{4}\right)\frac{\tau}{2}+\left(n+\frac{1}{2}\right)p_2+\frac{n}{2}\right)\\
&= e\left(\left(n-\left(p_1-\frac{1}{2}\right)\right)^2\frac{\tau}{2}+\left(n+\frac{1}{2}\right)p_2+\frac{n}{2}\right)\\
&= e\left(p_1p_2+\frac{p_1}{2}-\frac{1}{4}\right)e\left(\left(n-p_1+\frac{1}{2}\right)^2\frac{\tau}{2}+\left(n-p_1+\frac{1}{2}\right)\left(p_2+\frac{1}{2}\right)\right).
\end{aligned} \tag{2.75}$$

Thus,

$$f_{p_1,p_2}(\tau) = \frac{e\left(p_1p_2+\frac{p_1}{2}-\frac{p_2}{2}-\frac{1}{4}\right)}{\eta(\tau)}\sum_{n=-\infty}^{\infty} e\left(\left(n-p_1+\frac{1}{2}\right)^2\frac{\tau}{2}+\left(n-p_1+\frac{1}{2}\right)\left(p_2+\frac{1}{2}\right)\right) \tag{2.76}$$

$$= \frac{e\left(\left(p_1-\frac{1}{2}\right)\left(p_2+\frac{1}{2}\right)\right)\vartheta_{\frac{1}{2}+p_2,\frac{1}{2}-p_1}(\tau)}{\eta(\tau)}, \tag{2.77}$$

completing the proof of the proposition. □

This completes the proof of Theorem 1.4. Theorem 1.3 follows by specialisation of the variables.

2.4. **Proof of the Kronecker limit formula at $s=0$.** Before we can prove the Kronecker limit formula at $s=0$, we first need to prove Theorem 1.5, the functional equation for the definite zeta function.

Proof. Fix $r>0$, and split up the Mellin transform integral into two pieces,

$$\hat{\zeta}_{p,q}(\Omega,s) = \int_0^\infty \Theta_{p,q}(t\Omega)t^s\frac{dt}{t} \tag{2.78}$$

$$= \int_r^\infty \Theta_{p,q}(t\Omega)t^s\frac{dt}{t} + \int_0^r \Theta_{p,q}(t\Omega)t^s\frac{dt}{t}. \tag{2.79}$$

Replacing t by t^{-1}, and then using the transformation law for the definite theta function (part (3) of Proposition 2.12 of [5]), the second integral is

$$\int_0^r \Theta_{p,q}(t\Omega)t^s \frac{dt}{t} = \int_{r^{-1}}^{\infty} \Theta_{p,q}^{c_1,c_2}(t^{-1}\Omega)t^{-s}\frac{dt}{t} \tag{2.80}$$

$$= \int_{r^{-1}}^{\infty} \frac{e(p^\top q)}{\sqrt{\det(-it\Omega)}} \Theta_{-q,p}(-(t^{-1}\Omega)^{-1})t^{-s}\frac{dt}{t} \tag{2.81}$$

$$= \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^{\infty} \Theta_{-q,p}(t(-\Omega^{-1}))t^{\frac{g}{2}-s}\frac{dt}{t}. \tag{2.82}$$

Putting it all together, we have

$$\hat{\zeta}_{p,q}(\Omega, s) = \int_r^{\infty} \Theta_{p,q}(t\Omega)t^s \frac{dt}{t} + \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^{\infty} \Theta_{-q,p}(t(-\Omega^{-1}))t^{\frac{g}{2}-s}\frac{dt}{t}. \tag{2.83}$$

The Θ-functions in both integrals decay exponentially as $t \to \infty$, so the right-hand side converges for all $s \in \mathbb{C}$. The right-hand side is obviously analytic for all $s \in \mathbb{C}$, so we've analytically continued $\hat{\zeta}_{p,q}(\Omega, s)$ to an entire function of s. Finally, we must prove the functional equation. If we plug $\frac{g}{2} - s$ for s in eq. (2.83), factor out the coefficient of the second term, and switch the order of the two terms, we obtain

$$\hat{\zeta}_{p,q}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \left(\int_{r^{-1}}^{\infty} \Theta_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(t(-\Omega^{-1}))t^s \frac{dt}{t} - \frac{e(-p^\top q)}{\sqrt{\det(i\Omega^{-1})}} \int_r^{\infty} \Theta_{p,q}^{c_1,c_2}(t\Omega)t^{\frac{g}{2}-s}\frac{dt}{t} \right). \tag{2.84}$$

Reusing eq. (2.83) on $\hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(-\Omega^{-1}, s)$, and appealing to the fact that $\Theta_{p,q}^{c_1,c_2}(\Omega) = -\Theta_{-p,-q}^{c_1,c_2}(\Omega)$, we have

$$\hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}, s\right) = \int_{r^{-1}}^{\infty} \Theta_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(t(-\Omega^{-1}))t^s \frac{dt}{t} - \frac{e(-p^\top q)}{\sqrt{\det(i\Omega^{-1})}} \int_r^{\infty} \Theta_{p,q}^{c_1,c_2}(t\Omega)t^{\frac{g}{2}-s}\frac{dt}{t}. \tag{2.85}$$

The functional equation now follows from eq. (2.84) and eq. (2.85). □

We are now ready to prove Theorem 1.6.

Set $\left(\begin{smallmatrix}\tilde\omega_{11} & \tilde\omega_{12}\\ \tilde\omega_{12} & \tilde\omega_{22}\end{smallmatrix}\right) := -\Omega^{-1}$, $\tilde\tau_1 := -1/\tau_1$, and $\tilde\tau_2 := -1/\tau_2$. A direct computation shows that $\left(\begin{smallmatrix}\tilde\omega_{11} & \tilde\omega_{12}\\ \tilde\omega_{12} & \tilde\omega_{22}\end{smallmatrix}\right) = \frac{1}{\det(-i\Omega)}\left(\begin{smallmatrix}\omega_{22} & -\omega_{12}\\ -\omega_{12} & \omega_{11}\end{smallmatrix}\right)$, $\tilde\tau_1 = \frac{-\tilde\omega_{12}-\sqrt{\det(i\Omega^{-1})}}{\tilde\omega_{11}}$, and $\tilde\tau_2 = \frac{-\tilde\omega_{12}+\sqrt{\det(i\Omega^{-1})}}{\tilde\omega_{11}}$. Moreover,

$$f_{p_1,p_2}(-1/\tau) = \frac{e\left(\left(p_1-\frac12\right)\left(p_2+\frac12\right)\right)\vartheta_{\frac12+p_2,\frac12-p_1}(-1/\tau)}{\eta(-1/\tau)} \tag{2.86}$$

$$= \frac{e\left(\left(p_1-\frac12\right)\left(p_2+\frac12\right)\right)e\left(\left(\frac12+p_2\right)\left(\frac12-p_1\right)\right)\sqrt{-i\tau}\,\vartheta_{p_1-\frac12,p_2+\frac12}(\tau)}{\sqrt{-i\tau}\,\eta(\tau)} \tag{2.87}$$

$$= \frac{\vartheta_{p_1-\frac12,p_2+\frac12}(\tau)}{\eta(\tau)}. \tag{2.88}$$

Thus, using theorem 1.5,

$$\hat\zeta_{0,q}(\Omega,0) = \frac{1}{\sqrt{\det(-i\Omega)}}\hat\zeta_{-q,0}(-\Omega^{-1},1) \tag{2.89}$$

$$= \frac{-1}{\sqrt{\det(-i\Omega)}\sqrt{\det(i\Omega^{-1})}}\left(\left(\operatorname{Log} f_{1-q_1,1-q_2}\right)(-1/\tau_1) + \left(\operatorname{Log} f_{1-q_1,1-q_2}\right)(1/\tau_2)\right) \tag{2.90}$$

$$= -\left(\left(\operatorname{Log} g_{q_1,q_2}\right)(\tau_1) + \left(\operatorname{Log} g_{q_1,q_2}\right)(-\tau_2)\right), \tag{2.91}$$

where $g_{q_1,q_2}(\tau) = \frac{\vartheta_{\frac12-q_1,\frac32-q_2}(\tau)}{\eta(\tau)}$. This completes the proof of Theorem 1.6.

Proposition 1.7 follows by specialisation of the variables.

3. Kronecker limit formulas for indefinite zeta functions

At a high level, the method of proof in the indefinite case is the same—compute the Fourier series in ξ for an indefinite theta function with respect to an action by a one-parameter unipotent subgroup $\{T^\xi\}$ of $\mathbf{SL}_2(\mathbb{R})$. However, the details are quite different. We take Mellin transforms and specialise some variables earlier in the calculation than in the definite case. We must allow ξ to be a complex number later in the calculation and perform a fairly delicate contour integration. Unlike in the definite case, the Fourier coefficients of the indefinite theta are not elementary functions, which ultimately leads to a more complicated Kronecker limit formula.

Let $c_1, c_2 \in \mathbb{C}^2$ satisfying $\overline{c_j}^\top M c_j < 0$, and consider the indefinite theta $\Theta^{c_1,c_2}_{p,q}$ with characteristics $p, q \in \mathbb{R}^2$, defined in section 1.4. Let $t > 0$, $\Omega \in \mathcal{H}_2^{(1)}$, and $M = \operatorname{Im}(\Omega)$. Write the indefinite theta of $t\Omega$ as

$$\Theta^{c_1,c_2}_{p,q}(t\Omega) = \sum_{n\in\mathbb{Z}^2} \rho^{c_1,c_2}_{\operatorname{Im}(t\Omega)}(n+q)\, e\left(Q_\Omega(n+q)t + p^\top(n+q)\right) \tag{3.1}$$

$$= \sum_{n\in\mathbb{Z}^2} \rho^{c_1,c_2}_{M}\left((n+q)t^{1/2}\right) e\left(Q_\Omega(n+q)t + p^\top(n+q)\right), \tag{3.2}$$

where

$$\rho^{c_1,c_2}_M(v) = \mathcal{E}\left(\frac{c_2^\top M v}{\sqrt{-\frac12 c_2^\top M c_2}}\right) - \mathcal{E}\left(\frac{c_1^\top M v}{\sqrt{-\frac12 c_1^\top M c_1}}\right), \tag{3.3}$$

and

$$\mathcal{E}(z) = \int_0^z e^{-\pi u^2}\,du. \tag{3.4}$$

3.1. **Some integrals involving $\mathcal{E}(u)$.** We will now prove a few integral formulas that we will need.

Lemma 3.1. *Suppose that $\alpha, \beta \in \mathbb{C}$ satisfy $\mathrm{Re}\,(\alpha^2 - 2i\beta) > 0$. Then, using the standard branch of the square root function,*

$$\int_0^\infty \mathcal{E}(\alpha t^{1/2}) e(\beta t)\,dt = \frac{-\alpha}{4\pi i\beta\sqrt{\alpha^2 - 2i\beta}}. \tag{3.5}$$

Proof. By integration by parts,

$$\begin{aligned}
\int_0^\infty \mathcal{E}(\alpha t^{1/2}) e(\beta t)\,dt &= \frac{1}{2\pi i\beta}\int_0^\infty \mathcal{E}(\alpha t^{1/2})\,\frac{d\left(e(\beta t)\right)}{dt}\,dt && (3.6)\\
&= \frac{1}{2\pi i\beta}\left(\mathcal{E}(\alpha t^{1/2}) e(\beta t)\Big|_{t=0}^{\infty} - \int_0^\infty e^{-\pi\alpha^2 t}\frac{\alpha}{2} t^{-1/2} e(\beta t)\right) && (3.7)\\
&= \frac{-\alpha}{4\pi i\beta}\int_0^\infty \exp\left(-\left(\pi\alpha - 2\pi i\beta\right)t\right) t^{1/2}\,\frac{dt}{t} && (3.8)\\
&= \frac{-\alpha}{4\pi i\beta}\int_C \exp(-u)\left(\frac{u}{\pi\alpha^2 - 2\pi i\beta}\right)^{1/2}\,\frac{du}{u} && (3.9)\\
&= \frac{-\alpha}{4\pi^{3/2} i\beta\sqrt{\alpha^2 - 2i\beta}}\int_C e^{-u}u^{1/2}\,\frac{du}{u}, && (3.10)
\end{aligned}$$

where the contour C is a ray from the origin through the point $\alpha^2 - 2i\beta$. If $z \in \mathbb{C}$ with $x = \mathrm{Re}(z) > 0$, $s \in \mathbb{C}$ with $\sigma = \mathrm{Re}(s) > 0$, and $[z_1, z_2]$ denotes the oriented line segment from z_1 to z_2, then

$$\begin{aligned}
\lim_{N\to\infty}\int_{[0,Nz]} e^{-u}u^s\,\frac{du}{u} &= \lim_{N\to\infty}\left(\int_{[0,Nx]} e^{-u}u^s\,\frac{du}{u} + \int_{[Nx,Nz]} e^{-u}u^s\,\frac{du}{u}\right) && (3.11)\\
&= \Gamma(s) + \lim_{N\to\infty}\int_{[Nx,Nz]} e^{-u}u^s\,\frac{du}{u} && (3.12)\\
&= \Gamma(s) + \lim_{N\to\infty} O\left(e^{-Nx}N^\sigma\right) && (3.13)\\
&= \Gamma(s). && (3.14)
\end{aligned}$$

Thus, in particular, $\int_C e^{-u}u^{1/2}\,\frac{du}{u} = \Gamma\left(\frac{1}{2}\right) = \pi^{1/2}$. Plugging this into eq. (3.10) gives eq. (3.5). □

As usual, let $M = \mathrm{Im}(\Omega)$. Define the following auxiliary function, which will appear as a factor in the integral in the indefinite Kronecker limit formula.

Definition 3.2. For $v \in \mathbb{C}^2$ and $s \in \mathbb{C}$, set

$$\kappa_\Omega^c(v, s) := -\int_0^\infty \rho_M^c\left(vt^{1/2}\right) e\left(Q_\Omega(v)t\right) t^s\,\frac{dt}{t}. \tag{3.15}$$

Also, set

$$\kappa_\Omega^{c_1,c_2}(v,s) := \kappa_\Omega^{c_2}(v,s) - \kappa_\Omega^{c_1}(v,s) \tag{3.16}$$

$$= \int_0^\infty \rho_M^{c_1,c_2}\left(vt^{1/2}\right) e\left(Q_\Omega(v)t\right) t^s \, \frac{dt}{t}. \tag{3.17}$$

In the case $s=1$, we will leave out s and set $\kappa_\Omega^c(v) := \kappa_\Omega^c(v,1)$, $\kappa_\Omega^{c_1,c_2}(v) := \kappa_\Omega^{c_1,c_2}(v,1)$.

In particular,

Corollary 3.3. *Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} Mcc^\top M$. Note that $\Lambda_c \in \mathcal{H}_2^{(0)}$ by Lemma 3.6 of* [5]*. Then,*

$$\kappa_\Omega^c(v) = \frac{c^\top M v}{4\pi i \sqrt{-Q_M(c)} Q_\Omega(v) \sqrt{-2iQ_{\Lambda_c}(v)}}. \tag{3.18}$$

Proof. Follows from Lemma 3.1. □

The following lemma will be needed to evaluate certain integrals.

Lemma 3.4. *For any real number $\alpha \in \mathbb{R}$,*

$$\int_0^\infty \rho_M^{c_1,c_2}\left(v\alpha t^{1/2}\right) e\left(Q_\Omega(v)\alpha^2 t\right) t^s \, \frac{dt}{t} = -\frac{\operatorname{sgn}(\alpha)}{|\alpha|^{2s}} \kappa_\Omega^{c_1,c_2}(v,s). \tag{3.19}$$

Proof. Follows from the definition of $\kappa_\Omega^{c_1,c_2}(v,s)$. □

3.2. **Fourier series of a unipotent transform of an indefinite theta function.** Consider the function of $\xi \in \mathbb{R}$ (although ξ will be allowed to be complex later on) and $t \in \mathbb{R}_{\geq 0}$,

$$h(\xi,t) := \Theta_{\left(T^\xi\right)^\top p, T^{-\xi} q}^{T^{-\xi}c_1, T^{-\xi}c_2}\left(t\left(T^\xi\right)^\top \Omega T^\xi\right) \tag{3.20}$$

$$= \sum_{n\in\mathbb{Z}^2} \rho_\Omega^{c_1,c_2}\left(\left(T^\xi n + q\right) t^{1/2}\right) e\left(Q_\Omega(T^\xi n + q)t + p^\top(T^\xi n + q)\right). \tag{3.21}$$

Write this function as a Fourier series,

$$h(\xi,t) = \sum_{k=-\infty}^{\infty} b_k(t) e(k\xi). \tag{3.22}$$

We are ultimately interested in the Mellin transform of this function,

$$\hat{\zeta}_{\left(T^\xi\right)^\top p, T^{-\xi} q}^{T^{-\xi}c_1, T^{-\xi}c_2}\left(\left(T^\xi\right)^\top \Omega T^\xi, s\right) = \int_0^\infty h(\xi,t) t^s \, \frac{dt}{t} \tag{3.23}$$

$$= \sum_{k=-\infty}^{\infty} \beta_k(s) e(k\xi), \tag{3.24}$$

where, as we will show,

$$\beta_k(s) := \int_0^\infty b_k(t) t^s \, \frac{dt}{t}. \tag{3.25}$$

Express $\Omega = \left(\begin{smallmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{smallmatrix}\right)$, $n = \left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right)$, $p = \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)$. Write $h(\xi,t) = \sum_{n_2=-\infty}^{\infty} h_{n_2}(\xi,t) = h_0(\xi,t) + \tilde{h}(\xi,t)$, where $h_j(\xi,t)$ is the sum over the terms with $n_2 = j$, and $\tilde{h}(\xi,t)$ is the sum over all the terms where $n_2 \neq 0$.

Also, assume that $q_1 = q_2 = 0$.

First, calculate $h_0(\xi,t)$:

$$h_0(\xi,t) = \sum_{n_1=-\infty}^{\infty} \rho_\Omega^{c_1,c_2}\begin{pmatrix} n_1 t^{1/2} \\ 0 \end{pmatrix} e\left(\frac{1}{2}\omega_{11}n_1^2 t + p_1 n_1\right). \tag{3.26}$$

The $n_1 = 0$ term of this sum vanishes.

We write, for $n_2 \neq 0$,

$$\begin{aligned}
&\int_0^1 h_{n_2}(\xi,t)e(-k\xi)\,d\xi \\
&= \int_0^1 \sum_{n_1=-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} n_1+n_2\xi \\ n_2 \end{pmatrix} t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} n_1+n_2\xi \\ n_2 \end{pmatrix} t + p^\top \begin{pmatrix} n_1+n_2\xi \\ n_2 \end{pmatrix}\right) e(-k\xi)\,d\xi && (3.27)\\
&= \sum_{n_1=0}^{n_2-1} \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} n_1+n_2\xi \\ n_2 \end{pmatrix} t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} n_1+n_2\xi \\ n_2 \end{pmatrix} t + p^\top \begin{pmatrix} n_1+n_2\xi \\ n_2 \end{pmatrix}\right) e(-k\xi)\,d\xi && (3.28)\\
&= \sum_{n_1=0}^{n_2-1} \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} n_2\xi \\ n_2 \end{pmatrix} t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} n_2\xi \\ n_2 \end{pmatrix} t + p^\top \begin{pmatrix} n_2\xi \\ n_2 \end{pmatrix}\right) e\left(-k\left(\xi - \frac{n_1}{n_2}\right)\right)\,d\xi && (3.29)\\
&= \left(\sum_{n_1=0}^{n_2-1} e\left(\frac{kn_1}{n_2}\right)\right) \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2}\right) \\
&\qquad \cdot e\left(Q_\Omega\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2\right) e\left(-k\xi\right)\,d\xi. && (3.30)
\end{aligned}$$

The exponential sum $\sum_{n_1=0}^{n_2-1} e\left(\frac{kn_1}{n_2}\right)$ evaluates to $|n_2|$ if $n_2|k$, and to 0 otherwise. Thus, for all $k \in \mathbb{Z}$ (including $k = 0$),

$$\begin{aligned}
&\int_0^1 \tilde{h}(\xi,t)e(-k\xi)\,d\xi && (3.31)\\
&= \sum_{n_2|k} |n_2| \int_{-\infty}^{\infty} \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2}\right) e\left(Q_\Omega\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2\right) e\left(-k\xi\right)\,d\xi.
\end{aligned}$$

Recall that, by our convention, a sum over $n_2|k$ ranges over both positive and negative n_2 (and over all integers when $k = 0$).

3.3. **Shifting the contour vertically.** Fix a positive real number λ to be specified later. Let C^+ (C^-) be the contour consisting of the horizontal line $\mathrm{Im}(z) = \lambda$ ($\mathrm{Im}(z) = -\lambda$), oriented towards the right half-plane. For each $d_1, d_2 \in \mathbb{Z}$, $d_2 \neq 0$, let $C(d_1,d_2)$ be C^+ if

$d_1d_2 > 0$ or $d_1 = 0$ and $d_2 > 0$; let $C(d_1,d_2)$ be C^- if $d_1d_2 < 0$ or $d_1 = 0$ and $d_2 < 0$. The integrands in eq. (3.31) approach zero as $\mathrm{Re}(\xi) \to \pm\infty$, so we may rewrite this formula using contour integrals

$$\begin{aligned}&\int_0^1 \tilde{h}(\xi,t)e(-k\xi)\,d\xi \qquad (3.32)\\ &= \sum_{n_2 \mid k} |n_2| \int_{C\left(\frac{k}{n_2},n_2\right)} \rho_M^{c_1,c_2}\left(\begin{pmatrix}\xi\\1\end{pmatrix} n_2 t^{\frac{1}{2}}\right) e\left(Q_\Omega\begin{pmatrix}\xi\\1\end{pmatrix} n_2^2 t + p^\top \begin{pmatrix}\xi\\1\end{pmatrix} n_2\right) e\left(-k\xi\right)\,d\xi.\end{aligned}$$

3.4. **Taking Mellin transforms term-by-term.** To calculate the Mellin transform of $h_0(\xi,t)$, we need to check absolute convergence to justify reversing the order of summation/integration.

Proposition 3.5. *If $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, then*

$$\int_0^\infty \sum_{n_1=-\infty}^{\infty} \left|\rho_\Omega^{c_1,c_2}\begin{pmatrix} n_1 t^{1/2}\\ 0\end{pmatrix} e\left(\frac{1}{2}\omega_{11}n_1^2 t + p_1 n_1\right)\right| t^\sigma \frac{dt}{t} < \infty. \qquad (3.33)$$

Proof. We bound the integral as follows.

$$\begin{aligned}&\int_0^\infty \sum_{n_1=-\infty}^{\infty} \left|\rho_\Omega^{c_1,c_2}\begin{pmatrix} n_1 t^{1/2}\\ 0\end{pmatrix} e\left(\frac{1}{2}\omega_{11}n_1^2 t + p_1 n_1\right)\right| t^\sigma \frac{dt}{t} && (3.34)\\ &= \int_0^\infty \sum_{n_1=-\infty}^{\infty} \left|\rho_\Omega^{c_1,c_2}\begin{pmatrix} t^{1/2}\\ 0\end{pmatrix} e\left(\frac{1}{2}\omega_{11} t\right)\right| \left(\frac{t}{n_1^2}\right)^\sigma \frac{dt}{t} && (3.35)\\ &= \left(\sum_{n_1=-\infty}^{\infty} |n_1|^{-2\sigma}\right)\left(\int_0^\infty \left|\rho_\Omega^{c_1,c_2}\begin{pmatrix} t^{1/2}\\ 0\end{pmatrix} e\left(\frac{1}{2}\omega_{11} t\right)\right| t^\sigma \frac{dt}{t}\right) && (3.36)\\ &< \infty. && (3.37)\end{aligned}$$

The sum converges for $\sigma > \frac{1}{2}$, and the integral converges for $\sigma > 0$ (as the integrand approaches a constant at $t \to 0$ and decays exponentially as $t \to \infty$). ☐

Therefore, we can switch the sum and the integral, and by Lemma 3.1 and dropping the subscript on n_1,

$$\begin{aligned}\int_0^\infty h_0(\xi,t)t^s \frac{dt}{t} &= -\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{\mathrm{sgn}(n)e(p_1 n)}{|n|^{2s}} \kappa_\Omega^{c_1,c_2}\left(\begin{pmatrix}1\\0\end{pmatrix}, s\right) && (3.38)\\ &= -\left(\mathrm{Li}_{2s}(e(p_1)) - \mathrm{Li}_{2s}(e(-p_1))\right)\kappa_\Omega^{c_1,c_2}\left(\begin{pmatrix}1\\0\end{pmatrix}, s\right). && (3.39)\end{aligned}$$

Next, we're going to calculate the Mellin transform of $\tilde{h}(\xi,t)$. We need an absolute convergence result to justifty our calculation here, too.

Proposition 3.6. *Suppose $\sigma = \mathrm{Re}(s) > \frac{1}{2}$. Then,*

$$\sum_{k\in\mathbb{Z}} \sum_{\substack{n_2 \mid k \\ n_2 \neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2} \right) \cdot e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 \right) e(-k\xi) t^s \right| \frac{dt}{t} d\xi < \infty. \tag{3.40}$$

Proof. Let

$$K^\pm = \int_0^\infty \int_{C^\pm} \left| \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} t^{1/2} \right) e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} t \right) \right| t^\sigma \, d\xi \frac{dt}{t} \tag{3.41}$$

$$< \infty. \tag{3.42}$$

Set $K = \max\{K^+, K^-\}$. We have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \sum_{\substack{n_2 \mid k \\ n_2 \neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2} \right) \cdot e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t + p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 \right) e(-k\xi) t^s \right| \frac{dt}{t} d\xi \\
&= \sum_{k\in\mathbb{Z}} \sum_{\substack{n_2 \mid k \\ n_2 \neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2 t^{1/2} \right) e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} n_2^2 t \right) \right| \cdot e^{-2\pi\lambda k} t^\sigma \frac{dt}{t} d\xi &(3.43) \\
&= \sum_{k\in\mathbb{Z}} \sum_{\substack{n_2 \mid k \\ n_2 \neq 0}} \int_0^\infty \int_{C\left(\frac{k}{n_2}, n_2\right)} \left| \rho_M^{c_1,c_2}\left(\begin{pmatrix} \xi \\ 1 \end{pmatrix} t^{1/2} \right) e\left(Q_\Omega \begin{pmatrix} \xi \\ 1 \end{pmatrix} t \right) \right| \cdot e^{-2\pi\lambda k} \left(\frac{t}{n_2^2} \right)^\sigma \frac{dt}{t} d\xi &(3.44) \\
&\leq K \sum_{k\in\mathbb{Z}} \sum_{\substack{n_2 \mid k \\ n_2 \neq 0}} e^{-2\pi\lambda k} n_2^{-2\sigma} &(3.45) \\
&= K \sum_{d_1\in\mathbb{Z}} \sum_{d_2 \in \mathbb{Z}\setminus\{0\}} e^{-2\pi\lambda |d_1 d_2|} d_2^{-2\sigma} &(3.46) \\
&< \infty. &(3.47)
\end{aligned}$$

The proposition is proved. □

Now we may justify taking the Mellin transform of the Fourier series term-by-term. It follows from Proposition 3.6 that

$$\hat{\zeta}_{\left(T^{\xi}\right)^{\top} p, 0}^{T^{-\xi} c_1, T^{-\xi} c_2}\left(\left(T^{\xi}\right)^{\top} \Omega T^{\xi}, s\right) = \int_0^\infty h(\xi, t) t^s \frac{dt}{t} \tag{3.48}$$

$$= \sum_{k=-\infty}^{\infty} \beta_k(s) e(k\xi), \tag{3.49}$$

where $\beta_k(s) := \int_0^\infty b_k(t) t^s \frac{dt}{t}$. Define $\tilde{\beta}_k(s) := \int_0^\infty \tilde{b}_k(t) t^s \frac{dt}{t}$; then,

$$\beta_k(s) = \begin{cases} -\left(\mathrm{Li}_{2s}(e(p_1)) - \mathrm{Li}_{2s}(e(-p_1))\right) \kappa_\Omega^{c_1,c_2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, s\right) + \tilde{\beta}_0(s) & \text{if } k = 0, \\ \tilde{\beta}_k(s) & \text{if } k \neq 0. \end{cases} \tag{3.50}$$

Proposition 3.6 also implies that we can switch the order of integration to compute

$$\tilde{\beta}_k(s) = \int_0^\infty \int_0^1 \tilde{h}(\xi, t) e(-k\xi)\, d\xi\, t^s \frac{dt}{t} \tag{3.51}$$

$$= \sum_{n_2 \mid k} |n_2| \int_{C\left(\frac{k}{n_2}, n_2\right)} e\left(n_2 p^\top \begin{pmatrix} \xi \\ 1 \end{pmatrix} - k\xi\right) \left(-\operatorname{sgn}(n_2) |n_2|^{-2s} \kappa_\Omega^{c_1,c_2}(\xi, s)\right) d\xi \tag{3.52}$$

$$= -\sum_{n_2 \mid k} \frac{\operatorname{sgn}(n_2)}{|n_2|^{2s-1}} \int_{C\left(\frac{k}{n_2}, n_2\right)} e\left(n_2(p_1\xi + p_2) - k\xi\right) \kappa_\Omega^{c_1,c_2}(\xi, s)\, d\xi. \tag{3.53}$$

3.5. **Series manipulations.** In this subsection, we set $\xi = 0$ in eq. (3.49). We will manipulate the right-hand side of this equation to prove Theorem 1.19. First of all, we have

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, s) = \sum_{k=-\infty}^{\infty} \beta_k(s) \tag{3.54}$$

$$= -\left(\mathrm{Li}_{2s}(e(p_1)) - \mathrm{Li}_{2s}(e(-p_1))\right) \kappa_\Omega^{c_1,c_2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, s\right) + \sum_{k=-\infty}^{\infty} \tilde{\beta}_k(s). \tag{3.55}$$

We will rewrite the sum of the $\tilde{\beta}_k(s)$ using the substitution $(d_1, d_2) = (\frac{k}{n_2}, n_2)$. The following manipulation is legal by Proposition 3.6.

$$\sum_{k=-\infty}^{\infty} \tilde{\beta}_k(s) = -\sum_{k \in \mathbb{Z}} \sum_{\substack{n_2 \mid k \\ n_2 \neq 0}} \frac{\operatorname{sgn}(n_2)}{|n_2|^{2s-1}} \int_{C\left(\frac{k}{n_2}, n_2\right)} e\left(n_2(p_1\xi + p_2) - k\xi\right) \kappa_\Omega^{c_1,c_2}(\xi, s)\, d\xi \tag{3.56}$$

$$= -\sum_{d_1 \in \mathbb{Z}} \sum_{d_2 \in \mathbb{Z} \setminus \{0\}} \frac{\operatorname{sgn}(d_2)}{|d_2|^{2s-1}} \int_{C(d_1, d_2)} e\left(d_2(p_1\xi + p_2) - d_1 d_2 \xi\right) \kappa_\Omega^{c_1,c_2}(\xi, s)\, d\xi. \tag{3.57}$$

Split up the series into four pieces.

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\tilde{\beta}_k(s) = &-\sum_{d_1>0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^-} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1>0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&-\sum_{d_1\le 0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1\le 0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^-} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \qquad (3.58)\\
= &-\sum_{d_1>0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi \\
&+\sum_{d_1>0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&-\sum_{d_1\le 0}\sum_{d_2>0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left(-(d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1\le 0}\sum_{d_2<0}\frac{e(d_2p_2)}{|d_2|^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi \qquad (3.59)\\
= &-\sum_{d_1>0}\sum_{d_2>0}\frac{e(d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi \\
&+\sum_{d_1>0}\sum_{d_2>0}\frac{e(-d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1-p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&-\sum_{d_1\ge 0}\sum_{d_2>0}\frac{e(d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1+p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(\xi,s)\,d\xi \\
&+\sum_{d_1\ge 0}\sum_{d_2>0}\frac{e(-d_2p_2)}{d_2^{2s-1}}\int_{C^+} e\left((d_1+p_1)d_2\xi\right)\kappa_\Omega^{c_1,c_2}(-\xi,s)\,d\xi. \qquad (3.60)\end{aligned}$$

Now, move the contour integral outside the sums, and rewrite the series as

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\tilde{\beta}_k(s) = \int_{C+}\Bigg(&\sum_{d_2\ge 0}\frac{e(-p_2+p_1\xi)^{d_2}}{d_2^{2s-1}}\kappa_\Omega^{c_1,c_2}(-\xi,s) - \sum_{d_2\ge 0}\frac{e(p_2+p_1\xi)^{d_2}}{d_2^{2s-1}}\kappa_\Omega^{c_1,c_2}(\xi,s) \qquad (3.61)\\
&+\sum_{d_1>0}\sum_{d_2>0}\frac{1}{d_2^{2s-1}}\bigg(\Big(-e\left((d_1-p_1)\xi+p_2\right)^{d_2} + e\left((d_1+p_1)\xi-p_2\right)^{d_2}\Big)\kappa_\Omega^{c_1,c_2}(-\xi,s) \\
&+\Big(e\left((d_1-p_1)\xi-p_2\right)^{d_2} - e\left((d_1+p_1)\xi+p_2\right)^{d_2}\Big)\kappa_\Omega^{c_1,c_2}(\xi,s)\bigg)\Bigg)\,d\xi.\end{aligned}$$

Setting $s = 1$, we obtain

$$\sum_{k=-\infty}^{\infty} \tilde{\beta}_k(1) \tag{3.62}$$

$$= \int_{C+} \Bigg(- \log(1 - e(-p_2 + p_1\xi))\kappa_\Omega^{c_1,c_2} \begin{pmatrix} -\xi \\ 1 \end{pmatrix} + \log(1 - e(p_2 + p_1\xi))\kappa_\Omega^{c_1,c_2} \begin{pmatrix} \xi \\ 1 \end{pmatrix}$$

$$+ \sum_{d_1=1}^{\infty} \Bigg(\left(\log\left(1 - e\left((d_1 - p_1)\xi + p_2\right)\right) - \log\left(1 - e\left((d_1 + p_1)\xi - p_2\right)\right)\right) \kappa_\Omega^{c_1,c_2} \begin{pmatrix} -\xi \\ 1 \end{pmatrix}$$

$$\left(- \log\left(1 - e\left((d_1 - p_1)\xi - p_2\right)\right) + \log\left(1 - e\left((d_1 + p_1)\xi + p_2\right)\right)\right) \kappa_\Omega^{c_1,c_2} \begin{pmatrix} \xi \\ 1 \end{pmatrix} \Bigg) \Bigg) \, d\xi.$$

We want to write this sum of logarithms as a logarithm of a product, but there is the issue of the choice of branch. In order to make a clear choice, let

$$\varphi_{p_1,p_2}(\xi) := \left(1 - e\left(p_1\xi + p_2\right)\right) \prod_{d=1}^{\infty} \frac{1 - e\left((d + p_1)\xi + p_2\right)}{1 - e\left((d - p_1)\xi - p_2\right)} \tag{3.63}$$

for $\xi \in \mathcal{H}$. This is a function on the upper half-plane which is never zero, and the upper half-plane is simply connected, so it has a choice of continuous logarithm. Let $\left(\operatorname{Log} \varphi_{p_1,p_2}\right)(\xi)$ be the branch such that

$$\lim_{\xi \to i\infty} \left(\operatorname{Log} \varphi_{p_1,p_2}\right)(\xi) = \begin{cases} \log(1 - e(p_2)) & \text{if } p_1 = 0, \\ 0 & \text{if } p_1 \neq 0. \end{cases} \tag{3.64}$$

Here $\log(1 - e(p_2))$ is the standard principal branch. Thus,

$$\sum_{k=-\infty}^{\infty} \tilde{\beta}_k(1) = \int_{C+} \Bigg(- \left(\operatorname{Log} \varphi_{p_1,-p_2}\right)(\xi) \cdot \kappa_\Omega^{c_1,c_2} \begin{pmatrix} -\xi \\ 1 \end{pmatrix}$$

$$+ \left(\operatorname{Log} \varphi_{p_1,p_2}\right)(\xi) \cdot \kappa_\Omega^{c_1,c_2} \begin{pmatrix} \xi \\ 1 \end{pmatrix} \Bigg) \, d\xi. \tag{3.65}$$

Adding back the other piece of $\beta_0(1)$ into $\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = \sum_{k=-\infty}^{\infty} \beta_k(1)$, we obtain

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = - \left(\operatorname{Li}_2(e(p_1)) - \operatorname{Li}_2(e(-p_1))\right) \kappa_\Omega^{c_1,c_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.66}$$

$$+ \int_{C+} \Bigg(- \left(\operatorname{Log} \varphi_{p_1,-p_2}\right)(\xi) \cdot \kappa_\Omega^{c_1,c_2} \begin{pmatrix} -\xi \\ 1 \end{pmatrix} \tag{3.67}$$

$$+ \left(\operatorname{Log} \varphi_{p_1,p_2}\right)(\xi) \cdot \kappa_\Omega^{c_1,c_2} \begin{pmatrix} \xi \\ 1 \end{pmatrix} \Bigg) \, d\xi. \tag{3.68}$$

3.6. **Collapsing the contour onto the branch cuts.** We could declare ourselves done at this point. Equation (3.66) is a formula for $\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1)$, as we desired, and it appears very difficult to evaluate or simplify the contour integral in any way. However, eq. (3.66) is not a useful formula for computation because the integral converges slowly. The integrand decays polynomially as $\xi \to \pm\infty$ along the horocycle C^+.

We will obtain a Kronecker limit formula with rapid convergence by shifting the contour so that the integrand decays exponentially. In doing so, we will also split up the formula as a difference of a c_1-piece and a c_2-piece. The movement of the contour is shown in Section 3.6.

Let $\Lambda_c = \Omega - \frac{i}{Q_M(c)} M c c^\top M$ for $c = c_1, c_2$, as we did in Corollary 3.3. Factor the quadratic polynomial $\mathbb{Q}_{\Lambda_c}\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right)$ in ξ,

$$\mathbb{Q}_{\Lambda_c}\left(\begin{array}{c}\xi\\1\end{array}\right) = \alpha(c)(\xi - \tau_1(c))(\xi - \tau_2(c)). \tag{3.69}$$

Since $\Lambda_c \in \mathcal{H}_2^{(0)}$ by Lemma 3.6 of [5], we know by Lemma 2.4 that we may choose $\tau_1(c)$ to be in the upper half-plane and $\tau_2(c)$ in the lower half-plane.

The complex function $\xi \mapsto \kappa_\Omega^c\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right)$ has branch cuts along the vertical ray from $\tau_1(c)$ to $i\infty$ and the vertical ray from $\tau_2(c)$ to $-i\infty$. We check that this function is holomorphic away from these branch cuts. Since $\kappa_\Omega^c\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right)$ has simple poles at the roots $\xi = r_1, r_2$ of $Q_\Omega\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right) = 0$, we must check that the residues at the poles cancel when taking the difference $\kappa_\Omega^{c_1,c_2}\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right) = \kappa_\Omega^{c_2}\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right) - \kappa_\Omega^{c_1}\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right)$. We have

$$\begin{aligned}
&\operatorname{res}_{\xi\to r_1} \kappa_\Omega^c\left(\begin{array}{c}\xi\\1\end{array}\right)\\
&= \lim_{\xi\to r_1} (\xi - r_1) \frac{c^\top M\left(\begin{array}{c}\xi\\1\end{array}\right)}{2\pi i Q_\Omega\left(\begin{array}{c}\xi\\1\end{array}\right)\sqrt{\left(c^\top M\left(\begin{array}{c}\xi\\1\end{array}\right)\right)^2 - 2iQ_M(c)Q_\Omega\left(\begin{array}{c}\xi\\1\end{array}\right)}} &(3.70)\\
&= \lim_{\xi\to r_1} \frac{c^\top M\left(\begin{array}{c}\xi\\1\end{array}\right)}{\pi i\omega_{11}(\xi - r_2)\sqrt{\left(c^\top M\left(\begin{array}{c}\xi\\1\end{array}\right)\right)^2 - 2iQ_M(c)Q_\Omega\left(\begin{array}{c}\xi\\1\end{array}\right)}} &(3.71)\\
&= \frac{1}{\pi i\omega_{11}(r_1 - r_2)}, &(3.72)
\end{aligned}$$

and similarly, $\operatorname{res}_{\xi\to r_2} \kappa_\Omega^c\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right) = \frac{1}{\pi i\omega_{11}(r_2 - r_1)}$. These residues do not depend on c, so they cancel, and $\kappa_\Omega^{c_1,c_2}\left(\begin{smallmatrix}\xi\\1\end{smallmatrix}\right)$ is holomorphic at r_1 and r_2.

Move the countours of integration above the zeros of $Q_\Omega\left(\begin{smallmatrix}\pm\xi\\1\end{smallmatrix}\right)$. Now we may safely split up the integral into a term for c_1 and a term for c_2.

Now we retract the integral onto the branch cut. As $\xi = \pm\tau^\pm + \varepsilon$ and $\varepsilon \to 0$, the denominator of the integrand blows up like $\varepsilon^{1/2}$, so the integral converges. The integrand changes sign when we cross the branch cut. Thus, eq. (3.66) becomes

$$\hat{\zeta}_{p,0}^{c_1,c_2}(\Omega, 1) = I^+(c_2) - I^-(c_2) - I^+(c_1) + I^-(c_1), \tag{3.73}$$

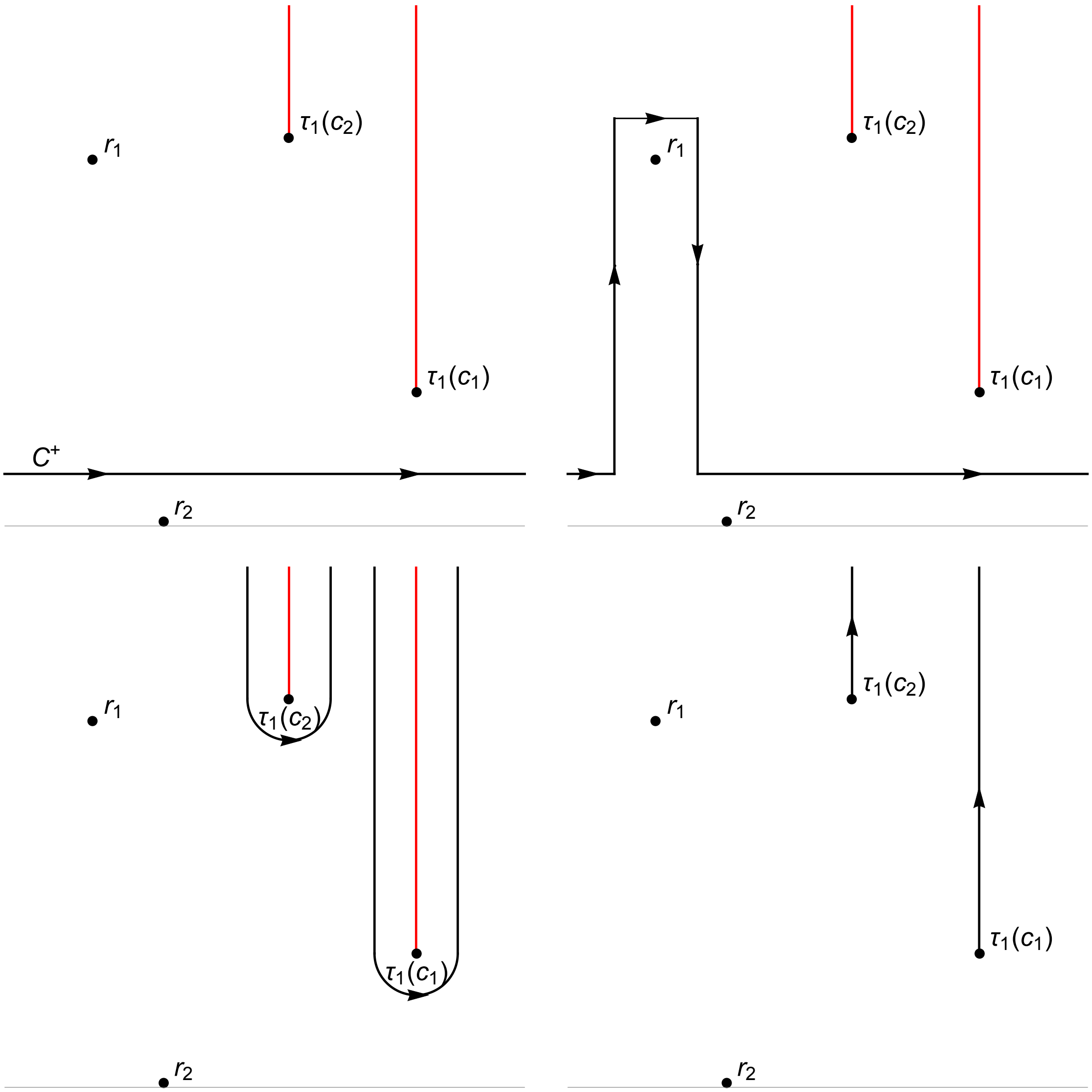

FIGURE 1. The contour C^+ is moved above the poles of $\kappa_\Omega^c \left(\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right)$, then collapsed onto branch cuts.

where

$$I^{\pm}(c) = -\operatorname{Li}_2(e(\pm p_1))\kappa_\Omega^c \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2i \int_0^\infty \left(\operatorname{Log} \varphi_{p_1, \pm p_2}\right)(\pm\tau^{\pm}(c) + it)\kappa_\Omega^c \begin{pmatrix} \pm(\tau^{\pm}(c) + it) \\ 1 \end{pmatrix} dt. \tag{3.74}$$

We have now proven Theorem 1.19. Theorem 1.20 follows by specializing the variables, setting $\Omega = iM$ and restricting to $c_1, c_2 \in \mathbb{R}^g$.

4. Example

We conclude with an example to show how to use the Kronecker limit formula for indefinite zeta functions to compute Stark units. This example was introduced in section 7.1 of [5].

Let $K = \mathbb{Q}(\sqrt{3})$, so $\mathbf{O}_K = \mathbb{Z}[\sqrt{3}]$, and let $\mathfrak{c} = 5\mathbf{O}_K$. The ray class group $\mathrm{Cl}_{\mathfrak{c}\infty_2} \cong \mathbb{Z}/8\mathbb{Z}$. The fundamental unit $\varepsilon = 2+\sqrt{3}$ is totally positive: $\varepsilon\varepsilon' = 1$. It has order 3 modulo 5: $\varepsilon^3 = 26+15\sqrt{3} \equiv 1 \pmod 5$. In this section, we use the Kronecker limit formula for indefinite zeta functions to compute $Z_I'(0)$, where I is the principal ray class of $\mathrm{Cl}_{\mathfrak{c}\infty_2}$.

Let $M = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -6 \end{smallmatrix}\right)$, $q = \left(\begin{smallmatrix} 1/5 \\ 0 \end{smallmatrix}\right)$, and $c_1 \in \mathbb{R}^2$ any column vector with the property that $c_1^\top M c_1 < 0$, such as $c_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. By Corollary 1.17 and the discussion in section 7.1 of [5], we have

$$Z_I'(0) = \hat{\zeta}_{0,q}^{c_1, P^3 c_1}(iM, 0). \tag{4.1}$$

Use theorem 1.12 (the functional equation for indefinite zeta functions) to write $Z_I'(0)$ in terms of an indefinite zeta value at $s=1$:

$$Z_I'(0) = \frac{1}{\sqrt{-12}} \hat{\zeta}_{-q,0}^{-iMc_1, -iMP^3c_1}(iM^{-1}, 1). \tag{4.2}$$

We have $Mc_1 = \left(\begin{smallmatrix} 0 \\ -6 \end{smallmatrix}\right) = -6c_1$. Let $\tilde{P} = MPM^{-1} = \left(\begin{smallmatrix} 2 & -1 \\ -3 & 2 \end{smallmatrix}\right)$. We may rescale the c_j without changing the value of the indefinite zeta function. Thus,

$$Z_I'(0) = \frac{-i}{2\sqrt{3}} \hat{\zeta}_{-q,0}^{c_1, \tilde{P}^3 c_1}(iM^{-1}, 1). \tag{4.3}$$

Now we want to use Theorem 1.20 to compute the right-hand side of eq. (4.3). If we try to do so directly, we obtain $\tilde{P}^3 c_1 = \left(\begin{smallmatrix} -15 \\ 26 \end{smallmatrix}\right)$ and $\kappa_\Omega^{\tilde{P}^3 c_1}\left(\left(\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right), 1\right) = \frac{6\sqrt{2}(45\xi+26)}{\pi(3\xi^2-1)\sqrt{4053\xi^2+4680\xi+1351}}$. The branch point of $\kappa_\Omega^{\tilde{P}^3 c_1}\left(\left(\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right), 1\right)$ in the upper half-plane is $\xi = \frac{-2340+i\sqrt{3}}{4053}$, which is very close to the real axis. That means we'd need to use about $\frac{\log(10)N}{\pi\sqrt{3}/4053} \approx 1700N$ terms in the product expansion of $\varphi_{p_1,p_2}(\xi)$ to compute $Z_I'(0)$ to N decimal places of accuracy. We technically have exponential decay, but it's not very useful.

It is much more practical to break up the zeta function into pieces. We can also improve the rate of convergence by choosing c_1 optimally; here, we will use $c = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$, $\tilde{P}c = \left(\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right)$.

$$\begin{aligned}
\hat{\zeta}_{-q,0}^{c_1,\tilde{P}^3c_1}(-\Omega^{-1},1) &= \hat{\zeta}_{-q,0}^{c,\tilde{P}^3c}(-\Omega^{-1},1) &(4.4)\\
&= \hat{\zeta}_{-q,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q,0}^{\tilde{P}c,\tilde{P}^2c}(-\Omega^{-1},1) + \hat{\zeta}_{-q,0}^{\tilde{P}^2c,\tilde{P}^3c}(-\Omega^{-1},1) &(4.5)\\
&= \hat{\zeta}_{-q_0,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q_1,0}^{c,\tilde{P}c}(-\Omega^{-1},1) + \hat{\zeta}_{-q_2,0}^{c,\tilde{P}c}(-\Omega^{-1},1), &(4.6)
\end{aligned}$$

where $q_0 = q = \frac{1}{5}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $q_1 = q = \frac{1}{5}\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$, and $q_2 = q = \frac{1}{5}\left(\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right)$ are obtained from the residues of $\varepsilon^0, \varepsilon^1, \varepsilon^2$ modulo 5.

Now, we have $\kappa_\Omega^c\left(\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right) = \frac{-3\sqrt{6}(x-1)}{\pi(3x^2-1)\sqrt{3x^2-3x+1}}$ and $\kappa_\Omega^{\tilde{P}c}\left(\begin{smallmatrix} \xi \\ 1 \end{smallmatrix}\right) = \frac{3\sqrt{6}(x+1)}{\pi(3x^2-1)\sqrt{3x^2+3x+1}}$, which is much more manageable. We computed the following values in Mathematica using 40 terms of the product expansion of φ_{p_1,p_2}.

$$\begin{aligned} I_0(\tilde{P}c) - I_0(c) \approx & -0.0592384391754448832935450798 7 \\ & + 3.6568783902031178613289385023 9i \end{aligned} \tag{4.7}$$

$$\begin{aligned} I_1(\tilde{P}c) - I_1(c) \approx & -1.3373302108594346921068501489 9 \\ & + 0.5247781252942466338755689916 7i \end{aligned} \tag{4.8}$$

$$\begin{aligned} I_2(\tilde{P}c) - I_2(c) \approx & \; 2.6405758727192221245648419060 7 \\ & + 0.5247781252942466338755689916 7i \end{aligned} \tag{4.9}$$

We now obtain

$$Z_I'(0) = \frac{-i}{2\sqrt{3}} \left(\hat{\zeta}^{c,\tilde{P}c}_{-q_0,0}(-\Omega^{-1},1) + \hat{\zeta}^{c,\tilde{P}c}_{-q_1,0}(-\Omega^{-1},1) + \hat{\zeta}^{c,\tilde{P}c}_{-q_2,0}(-\Omega^{-1},1) \right) \tag{4.10}$$

$$= \frac{1}{2\sqrt{3}} \operatorname{Im} \left((I_0(\tilde{P}c) - I_0(c)) + (I_1(\tilde{P}c) - I_1(c)) + (I_2(\tilde{P}c) - I_2(c)) \right) \tag{4.11}$$

$$\approx 1.358630653392208162595113082 30. \tag{4.12}$$

This agrees (to 30 decimal digits) with the computations described in section 7.1 of [5]. The conjectural Stark unit is $\exp(Z_I'(0)) \approx 3.890861713943079255337643959 62$. This number is appears to be the root of the polynomial

$$\begin{aligned} & x^8 - (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 + (225+130\sqrt{3})x^4 \\ & - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 - (8+5\sqrt{3})x + 1, \end{aligned} \tag{4.13}$$

which we have verified lies in the appropriate class field.

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