INDEFINITE ZETA FUNCTIONS

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Abstract. We define generalised zeta functions associated to indefinite quadratic forms of signature $(g-1,1)$—and more generally, to complex symmetric matrices whose imaginary part has signature $(g-1,1)$—and we investigate their properties. These indefinite zeta functions are defined as Mellin transforms of indefinite theta functions in the sense of Zwegers, which are in turn generalised to the Siegel modular setting. We prove an analytic continuation and functional equation for indefinite zeta functions. We also show that indefinite zeta functions in dimension 2 specialise to differences of ray class zeta functions of real quadratic fields, whose leading Taylor coefficients at $s=0$ are predicted by the Stark conjectures.

1. Introduction

Given a positive definite quadratic form $Q(x_1,\ldots x_g)$, it is possible to associate a zeta function $\zeta_Q(s)$, sometimes called the Epstein zeta function:

$$\zeta_Q(s) = \sum_{(n_1,\ldots,n_g)\in\mathbb{Z}^g\setminus\{0\}} \frac{1}{Q(n_1,\ldots,n_g)^s}. \tag{1.1}$$

The Epstein zeta function specializes to real analytic Eisenstein series (when $g=2$) and more specifically to class zeta functions of imaginary quadratic orders (when $g=2$ and Q has integral coefficients).

However, if Q is instead an indefinite quadratic form, the series in eq. (1.1) does not converge. One way to fix this issue it to restrict the sum to a closed subcone C of the double-cone of positivity $\{v\in\mathbb{R}^g : Q(v)>0\}$. This gives rise to a partial indefinite zeta function

$$\zeta_Q^C(s) = \sum_{(n_1,\ldots,n_g)\in C\cap\mathbb{Z}^g} \frac{1}{Q(n_1,\ldots,n_g)^s}. \tag{1.2}$$

However, unlike the Epstein zeta function, this partial zeta function does not satisfy a functional equation.

In this paper, we define a family of *completed indefinite zeta functions* that do satisfy a functional equation. In general, the completed indefinite zeta function $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega,s)$ is defined in terms of the following parameters:

- a complex symmetric (not necessarily Hermitian) matrix $\Omega=\Omega^\top=iM+N$, with $M=\mathrm{Im}(\Omega)$ having signature $(g-1,1)$;
- "characteristics" $p,q\in\mathbb{R}^g$;
- "cone parameters" $c_1,c_2\in\mathbb{C}^g$ satisfying the inequalities $\overline{c_j}^\top Mc_j<0$;
- a complex variable $s\in\mathbb{C}$.

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Due to invariance properties, $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s)$ remains well-defined with several of the parameters taken to be in quotient spaces:

- the characteristics on a torus, $p, q \in \mathbb{R}^g/\mathbb{Z}^g$;
- the cone parameters in complex projective space, $c_1, c_2 \in \mathbb{P}^{g-1}(\mathbb{C})$.

The completed indefinite zeta function satisfies a functional equation, given by the following theorem.

Theorem 1.1 (Analytic continuation and functional equation)**.** *The function $\hat{\zeta}_{a,b}^{c_1,c_2}(\Omega, s)$ may be analytically continued to an entire function on $\mathbb{C}$. It satisfies the functional equation*

$$\hat{\zeta}_{p,q}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}, s\right). \tag{1.3}$$

The completed indefinite zeta function is *not* generally given by a Dirichlet series. In the case of real cone parameters, it may be decomposed (up to Γ-factors) as a sum of an *incomplete indefinite zeta function* $\zeta_{p,q}^{c_1,c_2}(\Omega, s)$, which is a Dirichlet series, and correction terms $\xi_{p,q}^{c_j}(\Omega, s)$ that depend only on the cone parameters c_1 and c_2 separately.

Theorem 1.2 (Series decomposition)**.** *For real cone parameters $c_1, c_2 \in \mathbb{R}^g$, and $\operatorname{Re}(s) > 1$, the completed indefinite zeta function may be written as*

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \pi^{-s}\Gamma(s)\zeta_{p,q}^{c_1,c_2}(\Omega, s) - \pi^{-(s+\frac{1}{2})}\Gamma\left(s + \frac{1}{2}\right)\left(\xi_{p,q}^{c_2}(\Omega, s) - \xi_{p,q}^{c_1}(\Omega, s)\right), \tag{1.4}$$

where $M = \operatorname{Im}(\Omega)$,

$$\zeta_{p,q}^{c_1,c_2}(\Omega, s) = \frac{1}{2} \sum_{n \in \mathbb{Z}^g + q} \left(\operatorname{sgn}(c_1^\top Mn) - \operatorname{sgn}(c_2^\top Mn)\right) e\left(p^\top n\right) Q_{-i\Omega}(n)^{-s}, \tag{1.5}$$

and

$$\xi_{p,q}^{c}(\Omega, s) = \frac{1}{2} \sum_{\nu \in \mathbb{Z}^g + q} \operatorname{sgn}(c^\top Mn) e\left(p^\top n\right) \left(\frac{(c^\top Mn)^2}{Q_M(c)}\right)^{-s} \cdot {}_2F_1\left(s, s + \frac{1}{2}, s+1; \frac{2Q_M(c)Q_{-i\Omega}(n)}{(c^\top Mn)^2}\right). \tag{1.6}$$

Here, for any complex symmetric matrix Λ, $Q_\Lambda(v) = v^\top \Lambda v$ denotes the associated quadratic form; also, ${}_2F_1$ denotes a hypergeometric function (see eq. (7.1)). The summand in eq. (1.5) should always be interpreted as 0 when $\operatorname{sgn}(c_1^\top Mn) = \operatorname{sgn}(c_2^\top Mn)$; whenever it is nonzero, $\operatorname{Re}(Q_{-i\Omega}(n)) > 0$, and the complex power is interpreted as $Q_{-i\Omega}(n)^{-s} = \exp\left(-s \log\left(Q_{-i\Omega}(n)\right)\right)$ where $\log$ is the principal branch of the logarithm with a branch cut along the negative real axis.

The series defining the incomplete indefinite zeta function $\zeta_{p,q}^{c_1,c_2}(\Omega, s)$ is a variant of the partial indefinite zeta function 1.2, which may be seen by writing it as

$$\zeta_{p,q}^{c_1,c_2}(\Omega, s) = \sideset{}{^*}\sum_{n \in C^+ \cap (\mathbb{Z}^g + q)} e\left(p^\top n\right) Q_{-i\Omega}(n)^{-s} - \sideset{}{^*}\sum_{n \in C^- \cap (\mathbb{Z}^g + q)} e\left(p^\top n\right) Q_{-i\Omega}(n)^{-s}, \tag{1.7}$$

where $C^+ = \{v \in \mathbb{R}^g : \operatorname{sgn}(c_2^\top Mv) \leq 0 \leq \operatorname{sgn}(c_1^\top Mv)\}$ and $C^- = \{v \in \mathbb{R}^g : \operatorname{sgn}(c_1^\top Mv) \leq 0 \leq \operatorname{sgn}(c_2^\top Mv)\}$ are subcones of the two components of the double-cone of positivity of $Q_M(v)$, and the notation $\sum^*$ means that points on the boundary of the cone are weighted by $\frac{1}{2}$, except for $n = 0$, which is excluded.

The indefinite zeta function is defined as a Mellin transform of an *indefinite theta function* (literally, an *indefinite theta null with real characteristics*, see Definition 6.1 and the definitions in section 4). Indefinite theta functions were introduced by Sander Zwegers in his PhD thesis [20]. The indefinite theta functions introduced in this paper generalise Zwegers's work to the Siegel modular setting.

Our definition of indefinite zeta functions is in part motivated by an application to the computation of Stark units over real quadratic fields, which will be covered more thoroughly in a companion paper [13]. In special cases, an important symmetry, which we call *P-stability*, causes the ξ^{c_1} and ξ^{c_2} terms in eq. (1.4) to cancel, leaving a Dirichlet series $\zeta_{p,q}^{c_1,c_2}(\Omega, s)$. In the 2-dimensional case ($g = 2$), this Dirichlet series is a difference of two ray class zeta functions of an order in a real quadratic field.

Theorem 1.3 (Specialization to a ray class zeta function)**.** *Let K be a real quadratic number field, and let $\mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}$ denote the ray class group of $\mathcal{O}_K$ modulo $\mathfrak{c}\infty_1\infty_2$ (see eq. (8.1)). For each class $A \in \mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}$ and integral ideal $\mathfrak{b} \in A^{-1}$, there exists a real symmetric matrix M of signature $(1, 1)$, along with $c_1, c_2, q \in \mathbb{C}^2$, such that*

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s). \tag{1.8}$$

Here, $Z_A(s)$ is the differenced ray class zeta function associated to A (see Definition 8.2).

In the companion paper [13], we will prove a Kronecker limit formula for indefinite zeta functions in dimension 2, which specialises to an analytic formula for Stark units.

The (completed) incomplete zeta function could perhaps be called a "mock" zeta function, by analogy with mock modular forms, because it is "completed" to an object satisfying the appropriate symmetries by the addition of a non-Dirichlet (analogously, non-holomorphic) piece. This analogy is further supported the definition of the indefinite zeta function as a Mellin transform of an indefinite theta function. Zwegers used indefinite theta functions to construct harmonic weak Maass forms whose holomorphic parts are the mock theta functions of Ramanujan. Zwegers's work triggered an explosion of interest in mock modular forms, with applications to partition identities [4], quantum modular forms and false theta functions [8], period integrals of the j-invariant [6], sporadic groups [7], and quantum black holes [5]. Readers looking for additional exposition on these topics may also be interested in the book [3] (especially section 8.2) and lecture notes [17, 19].

This paper is organised as follows. In section 3, we review the theory of Riemann theta functions, which we extend to the indefinite case in section 4, generalising Chapter 2 of Zwegers's PhD thesis [20]. In section 5 and section 6, we define definite and indefinite zeta functions, respectively, and prove their analytic continuations and functional equations; in particular, we prove Theorem 1.1. In section 7, we prove a general series expansion for indefinite zeta functions, which is Theorem 1.2. In section 8, we prove that indefinite zeta functions restrict to differences of ray class zeta functions of real quadratic fields, which is Theorem 1.3, and we work through an example computation of a Stark unit using indefinite zeta functions.

2. Notation and convections

We list for reference the notational conventions used in this paper.

- $e(z) := \exp(2\pi i z)$ is the complex exponential, and this notation is used for $z \in \mathbb{C}$ not necessarily real.
- $\mathfrak{H} = \{\tau : \operatorname{Im} \tau > 0\}$ is the complex upper half-plane.
- Non-transposed vectors $v \in \mathbb{C}^g$ are always column vectors; the transpose $v^\top$ is a row vector.
- If Λ is a $g \times g$ matrix, then $\Lambda^\top$ is its transpose, and (when Λ is invertible) $\Lambda^{-\top}$ is a shorthand for $(\Lambda^{-1})^\top$.
- $Q_\Lambda(v)$ denotes the quadratic form $Q_\Lambda(v) = \frac{1}{2} v^\top \Lambda v$, where Λ is a $g \times g$ matrix, and v is a $g \times 1$ column vector.
- $f(c)|_{c=c_1}^{c_2} = f(c_2) - f(c_1)$, where f is any function taking values in an additive group.
- If $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ and f is a function of $\mathbb{C}^2$, we may write $f(v) = f \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ rather than $f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$.
- Unless otherwise specified, the logarithm $\log(z)$ is the standard principal branch with $\log(1) = 1$ and a branch cut along the negative real axis, and z^a means $\exp(a \log(z))$.
- Throughout the paper, Ω will be used to denote a $g \times g$ complex symmetric matrix. We will often need to express $\Omega = iM + N$ where M, N are real symmetric matrices. Matrices denoted by M and N will always have real entries, even when we do not say so explicitly.

3. Riemann theta functions

The definite theta function—or Riemann theta function—of dimension (or genus) g is a function of an elliptic parameter z and a modular parameter Ω. Riemann's theory generalizes the dimension 1 case of Jacobi theta functions. The elliptic parameter z lives in $\mathbb{C}^g$, but may (almost) be treated as an element of a complex torus $\mathbb{C}^g/\Lambda$, which happens to be an abelian variety. The parameter Ω is written as a complex $g \times g$ matrix and lives in the Siegal upper half-space $\mathfrak{H}_g$, whose definition imposes a condition on $M = \operatorname{Im}(\Omega)$.

3.1. Definitions and geometric context.

An *abelian variety* over a field K is a connected projective algebraic group; it follows from this definition that the group law of is abelian. (See [15] as a reference for all results mentioned in this discussion.) A *principal polarization* on an abelian variety A is an isomorphism between A and the dual abelian variety $A^\vee$. Over $K = \mathbb{C}$, every principally polarized abelian variety of dimension g is a complex torus of the form $A(\mathbb{C}) = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, where Ω is in the *Siegel upper half-space* (sometimes called the Siegel upper half-plane, although it is a complex manifold of dimension $\frac{g(g+1)}{2}$).

Definition 3.1. The *Siegel upper half-space* of genus g is defined to be the following open subset of the space $\mathbf{M}_g(\mathbb{C})$ of symmetric $g \times g$ complex matrices.

$$\mathfrak{H}_g^{(0)} = \mathfrak{H}_g = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^\top \text{ and } \operatorname{Im}(\Omega) \text{ is positive-definite}\}. \tag{3.1}$$

When $g = 1$, we recover the usual upper half-plane $\mathfrak{H}_1 = \mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$.

Definition 3.2. The *definite (Riemann) theta function* is, for $z \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$,

$$\Theta(z; \Omega) = \sum_{n \in \mathbb{Z}^g} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right). \tag{3.2}$$

Definition 3.3. When $g = 1$, the definite theta functions is called a *Jacobi theta function* and is denoted by $\vartheta(z, \tau) = \Theta([z], [\tau])$ for $z \in \mathbb{C}$ and $\tau \in \mathfrak{H}$.

It is a theorem that the complex structure on $A(\mathbb{C})$ determines the algebraic structure on $A_{\mathbb{C}}$. The functions $\Theta(z + t; \Omega)$ for representatives $t \in \mathbb{C}^g$ of 2-torsion points of $A(\mathbb{C})$ may be used to define an explicit holomorphic embedding of A as an algebraic locus in complex projective space. These shifts t are called *characteristics*. More details may be found in Chapter VI of [14], in particular pages 104–108.

The positive integer g is sometimes called the "genus" because the Jacobian $\mathrm{Jac}(C)$ of an algebraic curve of genus g is a principally polarized abelian variety of dimension g. Not all principally polarized abelian varieties are Jacobians of curves; the question of characterizing the locus of Jacobians of curves inside the moduli space of all principally polarized abelian varieties is known as the *Schottky problem*.

3.2. **A canonical square root.** On the Siegel upper half-space $\mathfrak{H}_g$, $\det(-i\Omega)$ has a canonical square root.

Lemma 3.4. *Let $\Omega \in \mathfrak{H}_g$. Then,*

$$\left(\int_{x\in\mathbb{R}^g} e\left(\frac{1}{2}x^\top \Omega x\right)\,dx\right)^2 = \frac{1}{\det(-i\Omega)}. \tag{3.3}$$

Proof. Equation (3.3) holds for Ω diagonal and purely imaginary by reduction to the one-dimensional case $\int_{-\infty}^{\infty} e^{-\pi a x^2}\,dx = \frac{1}{\sqrt{a}}$. Consequently, eq. (3.3) holds for any purely imaginary Ω by a change of basis, using spectral decomposition.

Consider the two sides of eq. (3.3) as holomorphic functions in $\frac{g(g+1)}{2}$ complex variables (the entries of Ω); they agree whenever those $\frac{g(g+1)}{2}$ variables are real. Because they are holomorphic, it follows by analytic continuation that they agree everywhere. □

Definition 3.5. Lemma 3.4 provides a canonical square root of $\det(-i\Omega)$:

$$\sqrt{\det(-i\Omega)} := \left(\int_{x\in\mathbb{R}^g} e\left(\frac{1}{2}x^\top \Omega x\right)\,dx\right)^{-1}. \tag{3.4}$$

Whenever we write "$\sqrt{\det(-i\Omega)}$" for $\Omega \in \mathfrak{H}_g$, we will be referring to this square root.

We will later need to use this square root to evaluate a shifted version of the integral that defines it.

Corollary 3.6. *Let $\Omega \in \mathfrak{H}_g$ and $c \in \mathbb{C}^g$. Then,*

$$\int_{x\in\mathbb{R}^g} e\left(\frac{1}{2}(x+c)^\top \Omega (x+c)\right)\,dx = \frac{1}{\sqrt{\det(-i\Omega)}}. \tag{3.5}$$

Proof. Fix Ω. The left-hand side of eq. (3.5) is constant for $c \in \mathbb{R}^g$, by Lemma 3.4. Because the left-hand side is holomorphic in c, it is in fact constant for all $c \in \mathbb{C}^g$. □

Note that, if $\Omega \in \mathfrak{H}_g$, then Ω is invertible and $-\Omega^{-1} \in \mathfrak{H}_g$. This is a special case of Proposition 4.3, which says, in particular, that $\mathfrak{H}_g$ is closed under the fractional linear transformation action of the symplectic group,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1} \text{ for } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}_g(\mathbb{R}). \tag{3.6}$$

In particular, $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \cdot \Omega = -\Omega^{-1}$.

The behavior of our canonical square root under the modular transformation $\Omega \mapsto -\Omega^{-1}$ is given by the following proposition.

Proposition 3.7. *If $\Omega \in \mathfrak{H}_g$, then $\sqrt{\det(-i\Omega)}\sqrt{\det(i\Omega^{-1})} = 1$.*

Proof. This follows from Definition 3.5 by plugging in $\Omega = iI$, because the function given by $\Omega \mapsto \sqrt{\det(-i\Omega)}\sqrt{\det(i\Omega^{-1})}$ is a continuous and takes values in $\{\pm 1\}$, and $\mathfrak{H}_g$ is connected. □

3.3. Transformation laws of definite theta functions.

Proposition 3.8. *The definite theta function for $z \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$ satisfies the following transformation law with respect to the z variable, for $a + \Omega b \in \mathbb{Z}^g + \Omega\mathbb{Z}^g$:*

$$\Theta(z + a + \Omega b, \Omega) = e\left(-\frac{1}{2}b^\top \Omega b - b^\top z\right)\Theta(z, \Omega). \tag{3.7}$$

Proof. The proof is a straightforward calculation. It may be found (using slightly different notation) as Theorem 4 on page 8–9 of [16]. □

Theorem 3.9. *The definite theta function for $z \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$ satisfies the following transformation laws with respect to the Ω variable, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, $B = B^\top$:*

(1) $\Theta(z; A^\top \Omega A) = \Theta(A^{-\top} z; \Omega)$.
(2) $\Theta(z; \Omega + 2B) = \Theta(z; \Omega)$.
(3) $\Theta(z; -\Omega^{-1}) = \frac{e\left(\frac{1}{2}z^\top \Omega z\right)}{\sqrt{\det(i\Omega^{-1})}}\Theta(\Omega z; \Omega)$.

Proof. The proof of (1) and (2) is a straightforward calculation. A more powerful version of this theorem, combining (1)–(3) into a single transformation law, appears as Theorem A on pages 86–87 of [16].

To prove (3), we apply the Poisson summation formula directly to the theta series. The Fourier transforms of the terms are given as follows.

$$\begin{aligned}
&\int_{\mathbb{R}^g} e\left(Q_\Omega(n) + n^\top z\right) e\left(-n^\top \nu\right)\, dn \\
&= \int_{\mathbb{R}^g} e\left(Q_\Omega(n) + n^\top (z - \nu)\right) && (3.8) \\
&= e\left(-Q_{-\Omega^{-1}}(z - \nu)\right) \int_{\mathbb{R}^g} e\left(Q_\Omega\left(n + \Omega^{-1}(z - \nu)\right)\right) && (3.9) \\
&= \frac{e\left(-Q_{-\Omega^{-1}}(z - \nu)\right)}{\sqrt{\det(-i\Omega)}}. && (3.10)
\end{aligned}$$

In the last line, we used Lemma 3.4 and Definition 3.5. Now, by the Poisson summation formula,

$$\Theta(z,\Omega) = \sum_{\nu\in\mathbb{Z}^g} \frac{e\left(-Q_{-\Omega^{-1}}(z-\nu)\right)}{\sqrt{\det(-i\Omega)}} \tag{3.11}$$

$$= \frac{e\left(Q_{-\Omega^{-1}}(z)\right)}{\sqrt{\det(-i\Omega)}} \sum_{\nu\in\mathbb{Z}^g} e\left(Q_{-\Omega^{-1}}(\nu) + \nu^\top\Omega^{-1}z\right) \tag{3.12}$$

$$= \frac{e\left(Q_{-\Omega^{-1}}(z)\right)}{\sqrt{\det(-i\Omega)}} \sum_{\nu\in\mathbb{Z}^g} e\left(Q_{-\Omega^{-1}}(\nu) - \nu^\top\Omega^{-1}z\right) \text{ (sending } \nu\mapsto-\nu) \tag{3.13}$$

$$= \frac{e\left(-\frac{1}{2}z^\top\Omega^{-1}z\right)}{\sqrt{\det(-i\Omega)}} \Theta\left(-\Omega^{-1}z, -\Omega^{-1}\right). \tag{3.14}$$

If Ω is replaced by $-\Omega^{-1}$, we obtain (3). □

As was mentioned, it is possible to combine all of the modular transformations into a single theorem describing the transformation of Ω under the action of $\mathbf{Sp}_{2g}(\mathbb{Z})$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega+B)(C\Omega+D)^{-1}. \tag{3.15}$$

This rule is already fairly complicated in dimension $g = 1$, where the tranformation law involves Dedekind sums. The general case is done in Chapter III of [16], with the main theorems stated on pages 86–90.

3.4. **Definite theta functions with characteristics.** There is another notation for theta functions, using "characteristics," and it will be necessary to state the transformation laws using this notation as well. We replace z with $z = p + \Omega q$ for real variables $p, q \in \mathbb{R}^g$. The reader is cautioned that the literature on theta functions contains conflicting conventions, and some authors may use notation identical to this one to mean something slightly different.

Definition 3.10. Define the *definite theta null with real characteristics* $p, q \in \mathbb{R}^g$, for $\Omega \in \mathfrak{H}_g$:

$$\Theta_{p,q}(\Omega) = e\left(\frac{1}{2}q^\top\Omega q + p^\top q\right)\Theta\left(p+\Omega q, \Omega\right). \tag{3.16}$$

The transformation laws for $\Theta_{p,q}(\Omega)$ follow directly from those for $\Theta(z,\Omega)$.

Proposition 3.11. *Let $\Omega \in \mathfrak{H}_g$ and $p, q \in \mathbb{R}^g$. The elliptic transformation law for the definite theta null with real characteristics is given by*

$$\Theta_{p+a,q+b}(\Omega) = e\left(a^\top(q+b)\right)\Theta_{p,q}(\Omega). \tag{3.17}$$

for $a, b \in \mathbb{Z}^g$.

Proposition 3.12. *Let $\Omega \in \mathfrak{H}_g$ and $p, q \in \mathbb{R}^g$. The modular transformation laws for the definite theta null with real characteristics are given as follows, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, and $B = B^\top$.*

(1) $\Theta_{p,q}(A^\top\Omega A) = \Theta_{A^{-\top}p,Aq}(\Omega)$.
(2) $\Theta_{p,q}(\Omega+2B) = e(-q^\top Bq)\Theta_{p+2Bq,q}(\Omega)$.
(3) $\Theta_{p,q}\left(-\Omega^{-1}\right) = \frac{e\left(p^\top q\right)}{\sqrt{\det(i\Omega^{-1})}}\Theta_{-q,p}(\Omega)$.

4. Indefinite theta functions

If we allow $\mathrm{Im}(\Omega)$ to be indefinite, the series expansion in eq. (3.2) no longer converges anywhere. We want to remedy this problem by inserting a variable coefficient into each term of the sum. In Chapter 2 of his PhD thesis [20], Sander Zwegers found—in the case when Ω is purely imaginary—a choice of coefficients that preserves the transformation properties of the theta function.

The results of this section generalize Zwegers's work by replacing Zwegers's indefinite theta function $\vartheta_M^{c_1,c_2}(z,\tau)$ by the indefinite theta function $\Theta_\Omega^{c_1,c_2}[f](z,\Omega)$. The function has been generalized in the following ways.

- Replacing τM for $\tau \in \mathfrak{H}$ and $M \in M_g(\mathbb{R})$ real symmetric in of signature $(g-1,1)$ by $\Omega \in \mathfrak{H}_g^{(1)}$. (Adds $\frac{g(g+1)}{2} - 1$ real dimensions.)
- Allowing c_1, c_2 to be complex. (Adds $2g-2$ real dimensions.)
- Allowing a test function $f(u)$, which must be specialized to $f(u) = 1$ for all the modular transformation laws to hold.

One motivation for introducing a test function f is to find transformation laws for a more general class of test functions (e.g., polynomials). We may investigate the behaviour of test functions under modular transformations in future work. However, for the purpose of this paper, only the cases $u \mapsto |u|^r$ will be relevant.

4.1. The Siegel intermediate half-space.

Definition 4.1. If $M \in \mathbf{GL}_g(\mathbb{R})$ and $M = M^\top$, the *signature* of M (or of the quadratic form Q_M) is a pair (j,k), where j is the number of positive eigenvalues of M, and k is the number of negative eigenvalues (so $j+k=g$).

Definition 4.2. For $0 \le k \le g$, we define the *Siegel intermediate half-space* of genus g and index k to be

$$\mathfrak{H}_g^{(k)} = \{\Omega \in \mathbf{M}_g(\mathbb{C}) : \Omega = \Omega^\top \text{ and } \mathrm{Im}(\Omega) \text{ has signature } (g-k,k)\}. \tag{4.1}$$

We call a complex torus of the form $T_\Omega = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$ for $\Omega \in \mathfrak{H}_g^{(k)}$, $k \neq 0, g$, an *intermediate torus.*

Intermediate tori are usually *not* algebraic varieties. An example of intermediate tori in the literature are the intermediate Jacobians of Griffiths [9, 10, 11]. Intermediate Jacobians generalize Jacobians of curves, which are abelian varieties, but those defined by Griffiths are usually not algebraic. (In contrast, the intermediate Jacobians defined by Weil [18] are algebraic.)

The symplectic group $\mathbf{Sp}_{2g}(\mathbb{R})$ acts on the set of $g \times g$ complex symmetric matrices by the fractional linear transformation action,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}. \tag{4.2}$$

Proposition 4.3. *If* $\Omega \in \mathfrak{H}_g^{(k)}$ *and* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}_{2g}(\mathbb{R})$*, then* $(A\Omega+B)(C\Omega+D)^{-1} \in \mathfrak{H}_g^{(k)}$*. Moreover, the* $\mathfrak{H}_g^{(k)}$ *are the open orbits of the* $\mathbf{Sp}_{2g}(\mathbb{R})$*-action on the set of* $g \times g$ *complex symmetric matrices.*

Proof. Trivial for $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$. For $\begin{pmatrix} A^\top & 0 \\ 0 & A^{-1} \end{pmatrix}$, this is Sylvester's law of inertia. For $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, we have $\operatorname{Im}(-\Omega^{-1}) = \frac{1}{2i}(-\Omega^{-1} + \overline{\Omega}^{-1}) = \frac{1}{2i}\overline{\Omega}^{-1}(-\overline{\Omega} + \Omega)\Omega^{-1} = \overline{\Omega}^{-1}\operatorname{Im}(\Omega)\Omega^{-1} = \left(\overline{\Omega}^{-1}\right)^\top \operatorname{Im}(\Omega)\Omega^{-1}$, so $\operatorname{Im}(-\Omega^{-1})$ and $\operatorname{Im}(\Omega)$ have the same signature. These three types of matrices generate $\mathbf{Sp}_{2g}(\mathbb{R})$.

Now suppose $\Omega_1, \Omega_2 \in \mathfrak{H}_g^{(k)}$. There exists a matrix $A \in \mathbf{GL}_g(\mathbb{R})$ such that $A^\top \operatorname{Im}(\Omega_1) A = \operatorname{Im}(\Omega_2)$. For an appropriate choice of real symmetric $B \in \mathbf{M}_g(\mathbb{R})$, we thus have $A^\top \Omega_1 A + B = \Omega_2$. That is,

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} A^\top & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \Omega_1 = \Omega_2, \tag{4.3}$$

so Ω_1 and Ω_2 are in the same $\mathbf{Sp}_{2g}(\mathbb{R})$-orbit.

Thus, the $\mathfrak{H}_g^{(k)}$ are the open orbits of the $\mathbf{Sp}_{2g}(\mathbb{R})$-action on the set of $g \times g$ symmetric complex matrices. □

4.2. **More canonical square roots.** From now on, we will focus on the case of index $k = 1$, which is signature $(g-1, 1)$. The construction of modular theta series for $k \geq 2$ utilizes higher-order error functions arising in string theory [1]. More research is needed to develop the higher index theory in the Siegel modular setting.

Lemma 4.4. *Let M be a real symmetric matrix of signature $(g-1,1)$. On the region $R_M = \{z \in \mathbb{C}^g : \overline{z}^\top M z < 0\}$, there is a canonical choice of holomorphic function $g(z)$ such that $g(z)^2 = -z^\top M z$.*

Proof. By Sylvester's law of inertia, there is some $P \in \mathbf{GL}_g^+(\mathbb{R})$ (i.e., with $\det(P) > 0$) such that $M = P^\top J P$, where

$$J = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{4.4}$$

The region $S = \{(z_2, \dots, z_g) \in \mathbb{C}^{g-1} : |z_2|^2 + \cdots |z_g|^2 < 1\}$ is simply connected (as it is a solid ball) and does not intersect $\{(z_2, \dots, z_g) \in \mathbb{C}^{g-1} : z_2^2 + \cdots + z_g^2 = 1\}$ (because, if it did, we'd have $1 = \left|z_2^2 + \cdots + z_g^2\right| \leq |z_2|^2 + \cdots |z_g|^2 < 1$, a contradiction). Thus, there exists a unique continuous branch of the function $\sqrt{1 - z_2^2 - \cdots - z_g^2}$ on S sending $(0, \dots, 0) \mapsto 1$; this function is also holomorphic. For $z \in R_J$, define

$$g_J(z) := z_1 \sqrt{1 - \left(\frac{z_2}{z_1}\right)^2 - \cdots - \left(\frac{z_g}{z_1}\right)^2}. \tag{4.5}$$

This g_J is holomorphic and satisfies $g_J(z)^2 = -z^\top Jz$, $g_J(\alpha z) = \alpha g_J(z)$, and $g_J(e_1) = 1$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.6}$$

Conversely, if we have a continuous function $g(z)$ satisfying $g(z)^2 = -z^\top Jz$ and $g(e_1) = 1$, it follows that $g(\alpha z) = \alpha g(z)$, and thus $g(z) = g_J(z)$.

Now, we'd like to define $g_M(z) := g_J(Pz)$, so that we have $g_M(z)^2 = -z^\top Mz$. We need to check that this definition does not depend on the choice of P. Suppose $M = P_1^\top JP_1 = P_2^\top JP_2$ for $P_1, P_2 \in \mathbf{GL}_g^+(\mathbb{R})$. So $J = \left(P_2P_1^{-1}\right)^\top J \left(P_2P_1^{-1}\right)$, that is, $P_2P_1^{-1} \in \mathbf{O}(g-1,1)$. But $\det(P_2P_1^{-1}) = \det(P_2)\det(P_1)^{-1} > 0$, so, in fact, $P_2P_1^{-1} \in \mathbf{SO}(g-1,1)$.

For any $Q \in \mathbf{SO}(g-1,1)$, we have $g_J(Qe_1)^2 = 1$. The function $Q \mapsto g_J(Qe_1)$ must be either the constant 1 or the constant -1, because $\mathbf{SO}(g-1,1)$ is connected. Since $g_J(e_1) = 1$ ($Q = I$), we have $g_J(Qe_1) = 1$ for all $Q \in SO(1, g-1)$. The function $z \mapsto g_J(Qz)$ is a continuous square root of $-z^\top Jz$ sending e_1 to 1, so $g_J(Qz) = g_J(z)$. Taking $Q = P_2P_1^{-1}$ and replacing z with P_1z, we have $g_J(P_2z) = g_J(P_1z)$, as desired. □

Definition 4.5. If M is a real symmetric matrix of signature $(g-1,1)$, we will write $\sqrt{-z^\top Mz}$ for the function $g_M(z)$ in Lemma 4.4. We may also use similar notation, such as $\sqrt{-\frac{1}{2}z^\top Mz} := \frac{1}{\sqrt{2}}\sqrt{-z^\top Mz}$.

Lemma 4.6. *Suppose M is a real symmetric matrix of signature $(g-1,1)$, and $c \in \mathbb{C}^g$ such that $\overline{c}^\top Mc < 0$. Then, $M + M\operatorname{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1} cc^\top\right) M$ is well-defined (that is, $c^\top Mc \neq 0$) and positive definite.*

Proof. Because M has signature $(g-1,1)$ and $\overline{c}^\top Mc < 0$,

$$\left(\overline{c}^\top Mc\right)^2 - \left|c^\top Mc\right|^2 = \det\begin{pmatrix} \overline{c}^\top Mc & c^\top Mc \\ \overline{c}^\top M\overline{c} & c^\top M\overline{c} \end{pmatrix} < 0. \tag{4.7}$$

Thus, $\left|c^\top Mc\right| > \left(\overline{c}^\top Mc\right)^2 > 0$, so $c^\top Mc \neq 0$ and $M + M\operatorname{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1} cc^\top\right) M$ is well defined. Let

$$A = M + M\operatorname{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1} cc^\top\right) M \tag{4.8}$$

$$= M - M\left(c^\top Mc\right)^{-1} cc^\top M - M\left(\overline{c}^\top M\overline{c}\right)^{-1} \overline{c}\overline{c}^\top M. \tag{4.9}$$

On the $(g-1)$-dimensional subspace $W = \{w \in \mathbb{C}^g : \overline{c}^\top Mw = 0\}$, the sesquilinear form $w \mapsto \overline{w}^\top Mw$ is positive definite; this follows from the fact that $\overline{c}^\top Mc < 0$, because M has signature $(g-1,1)$. For nonzero $w \in W$,

$$\overline{w}^\top Aw = \overline{w}^\top Mw - (c^\top Mc)^{-1}(\overline{w}^\top Mc)(c^\top Mw) - (\overline{c}^\top M\overline{c})^{-1}(\overline{w}^\top M\overline{c})(\overline{c}^\top Mw) \tag{4.10}$$

$$= \overline{w}^\top Mw - (c^\top Mc)^{-1}(0)(c^\top Mw) - (\overline{c}^\top M\overline{c})^{-1}(\overline{w}^\top M\overline{c})(0) \tag{4.11}$$

$$= \overline{w}^\top Mw > 0. \tag{4.12}$$

Moreover,

$$c^\top Aw = c^\top Mw - (c^\top Mc)^{-1}(c^\top Mc)(c^\top Mw) - (\overline{c}^\top M\overline{c})^{-1}(c^\top M\overline{c})(\overline{c}^\top Mw) \tag{4.13}$$
$$= c^\top Mw - c^\top Mw - (\overline{c}^\top M\overline{c})^{-1}(c^\top M\overline{c})(0) \tag{4.14}$$
$$= 0, \tag{4.15}$$

and

$$c^\top A\overline{c} = c^\top M\overline{c} - (c^\top Mc)^{-1}(c^\top Mc)(c^\top M\overline{c}) - (\overline{c}^\top M\overline{c})^{-1}(c^\top M\overline{c})(\overline{c}^\top M\overline{c}) \tag{4.16}$$
$$= c^\top M\overline{c} - c^\top M\overline{c} - c^\top M\overline{c} \tag{4.17}$$
$$= -c^\top M\overline{c} \tag{4.18}$$
$$= -\overline{c}^\top Mc > 0. \tag{4.19}$$

We have now shown that A is positive definite, as it is positive definite on subspaces W and $\mathbb{C}\overline{c}$, and these subspaces span $\mathbb{C}^g$ and are perpendicular with respect to A. □

Lemma 4.7. *Let $\Omega = N + iM$ be an invertible complex symmetric $g \times g$ matrix. Consider $c \in \mathbb{C}^g$ such that $\overline{c}^\top Mc < 0$. The following identities hold:*

(1) $M\Omega^{-1} = \overline{\Omega}\,\mathrm{Im}\,(-\Omega^{-1})$.
(2) $M - 2iM\Omega^{-1}M = \overline{\Omega}\,\mathrm{Im}\,(-\Omega^{-1})\,\overline{\Omega}$.
(3) $\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right) = \det(-i\Omega)\frac{c^\top\overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}c}{c^\top Mc}$.

Proof. Proof of (1):

$$M\Omega^{-1} = \frac{1}{2i}(\Omega - \overline{\Omega})\Omega^{-1} \tag{4.20}$$
$$= \frac{1}{2i}(I - \overline{\Omega}\Omega^{-1}) \tag{4.21}$$
$$= \overline{\Omega}\frac{1}{2i}(\overline{\Omega}^{-1} - \Omega^{-1}) \tag{4.22}$$
$$= \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right). \tag{4.23}$$

Proof of (2):

$$M - 2iM\Omega^{-1}M = M\Omega^{-1}\left(\Omega - 2iM\right) \tag{4.24}$$
$$= \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\left(\Omega - (\Omega - \overline{\Omega})\right) \text{ using (1)} \tag{4.25}$$
$$= \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}. \tag{4.26}$$

Proof of (3): Note that $\det(I+A) = 1 + \mathrm{Tr}(A)$ for any rank 1 matrix A. Thus,

$$\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right) \tag{4.27}$$
$$= \det(-i\Omega)\det\left(I + \frac{2i}{c^\top Mc}(\Omega Mc)(Mc)^\top\right) \tag{4.28}$$
$$= \det(-i\Omega)\left(1 + \mathrm{Tr}\left(\frac{2i}{c^\top Mc}(\Omega Mc)(Mc)^\top\right)\right) \tag{4.29}$$
$$= \det(-i\Omega)\left(1 + \left(\frac{2i}{c^\top Mc}c^\top M\Omega^{-1}Mc\right)\right) \tag{4.30}$$
$$= \det(-i\Omega)\frac{-c^\top\left(M - 2iM\Omega^{-1}M\right)c}{-c^\top Mc} \tag{4.31}$$
$$= \det(-i\Omega)\frac{-(\overline{\Omega}c)^\top\,\mathrm{Im}\left(-\Omega^{-1}\right)(\overline{\Omega}c)}{-c^\top Mc}, \tag{4.32}$$

using (2) in the last step. □

Definition 4.8 (Canonical square root)**.** If $\Omega \in \mathfrak{H}_g^{(1)}$, then we define $\sqrt{\det(-i\Omega)}$ as follows. Write $\Omega = N + iM$ for $N, M \in \mathbf{M}_g(\mathbb{R})$, and choose any c such that $\overline{c}^\top Mc < 0$. By Lemma 4.6, $M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1}cc^\top\right)M$ is positive definite. Write $M + M\,\mathrm{Re}\left(\left(-\frac{1}{2}c^\top Mc\right)^{-1}cc^\top\right)M = \mathrm{Im}\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)$. By part (3) of Lemma 4.7,

$$\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right) = \det(-i\Omega)\frac{-(\overline{\Omega}c)^\top\,\mathrm{Im}\left(-\Omega^{-1}\right)(\overline{\Omega}c)}{-c^\top Mc}. \tag{4.33}$$

We can thus define $\sqrt{\det(-i\Omega)}$ as follows:

$$\sqrt{\det(-i\Omega)} := \frac{\sqrt{-c^\top Mc}\sqrt{\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right)}}{\sqrt{-(\overline{\Omega}c)^\top\,\mathrm{Im}\left(-\Omega^{-1}\right)(\overline{\Omega}c)}}, \tag{4.34}$$

where the square roots on the RHS are as defined in Definition 3.5 and Definition 4.5. This definition does not depend on the choice of c because the set $\{c \in \mathbb{C}^g : \overline{c}^\top Mc < 0\}$ is connected.

4.3. **Definition of indefinite theta functions.**

Definition 4.9. For any complex number α and any entire test function f, define the *incomplete Gaussian transform*

$$\mathcal{E}_f(\alpha) = \int_0^\alpha f(u)e^{-\pi u^2}\,du, \tag{4.35}$$

where the integral may be taken along any contour from 0 to α. In particular, for the constant functions $\mathbb{1}(u) = 1$, set

$$\mathcal{E}(\alpha) := \mathcal{E}_{\mathbb{1}}(\alpha) = \int_0^\alpha e^{-\pi u^2}\,du = \frac{\alpha}{2|\alpha|}\int_0^{|\alpha|^2} t^{-1/2}e^{-\pi(\alpha/|\alpha|)^2 t}\,dt. \tag{4.36}$$

When α is real, define $\mathcal{E}_g(\alpha)$ for an arbitrary continuous test function f:

$$\mathcal{E}_f(\alpha) = \int_0^\alpha f(u)e^{-\pi u^2}\,du. \tag{4.37}$$

Definition 4.10. Define the *indefinite theta function attached to the test function* f to be

$$\Theta^{c_1,c_2}[f](z;\Omega) = \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n + n^\top z\right), \tag{4.38}$$

where $\Omega \in \mathfrak{H}_g^{(1)}$, $z \in \mathbb{C}^g$, $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top Mc_1 < 0$, $\overline{c_2}^\top Mc_2 < 0$, and $f(\xi)$ is a continuous function of one variable satisfying the growth condition $\log|f(\xi)| = o\left(|\xi|^2\right)$. If the c_j are not both real, also assume that f is entire.

Also define the *indefinite theta function* $\Theta^{c_1,c_2}(z;\Omega) := \Theta^{c_1,c_2}[\mathbb{1}](z;\Omega)$.

The function $\Theta^{c_1,c_2}(z;\Omega) = \Theta^{c_1,c_2}[\mathbb{1}](z;\Omega)$ is the function we are most interested in, because it will turn out to satisfy a symmetry in $\Omega \mapsto -\Omega^{-1}$. We will also show that the functions $\Theta^{c_1,c_2}[u \mapsto |u|^r](z;\Omega)$ are equal (up to a constant) for certain special values of the parameters.

Before we can prove the transformation laws of our theta functions, we must show that the series defining them converges.

Proposition 4.11. *The indefinite theta series attached to f (eq. (4.38)) converges absolutely and uniformly for $z \in \mathbb{R}^g + iK$, where K is a compact subset of $\mathbb{R}^g$ (and for fixed Ω, c_1, c_2, and f).*

Proof. Let $M = \operatorname{Im}\Omega$. We may multiply c_1 and c_2 by any complex scalar without changing the terms of the series eq. (4.38), so we may assume without loss of generality that $\operatorname{Re}(\overline{c_1}^\top Mc_2) < 0$.

For $\lambda \in [0,1]$, define the vector $c(\lambda) = (1-\lambda)c_1 + \lambda c_2$ and the real symmetric matrix $A(\lambda) := M + M\operatorname{Re}\left(\left(-\frac{1}{2}c(\lambda)^\top Mc(\lambda)\right)^{-1} c(\lambda)c(\lambda)^\top\right)M$. Note that $\overline{c(\lambda)}^\top Mc(\lambda) = (1-\lambda)^2\overline{c_1}^\top Mc_1 + 2\lambda(1-\lambda)\operatorname{Re}(\overline{c_1}^\top Mc_2) + \lambda^2\overline{c_2}^\top Mc_2 < 0$ because each term is negative (except when $\lambda = 0$ or 1, in which case one term is negative and the others are zero). By Lemma 4.6, $A(\lambda)$ is well-defined and positive definite for each $\lambda \in [0,1]$.

Consider $(x,\lambda) \mapsto x^\top A(\lambda)x$ as a positive real-valued continuous function on the compact set that is the product of the unit ball $\{x^\top x = 1\}$ and the interval $[0,1]$. It has a global minimum $\varepsilon > 0$.

The parametrization $\gamma : [0,1] \to \mathbb{C}$, $\gamma(\lambda) := \frac{c(\lambda)^\top(Mn+y)}{\sqrt{-\frac{1}{2}c(\lambda)^\top Mc(\lambda)}}$, defines a countour from $\frac{c_1^\top(Mn+y)}{\sqrt{-\frac{1}{2}c_1^\top Mc_1}}$ to $\frac{c_2^\top(Mn+y)}{\sqrt{-\frac{1}{2}c_2^\top Mc_2}}$, so that

$$\mathcal{E}_f\left(\frac{c^\top(Mn+y)}{-\frac{1}{2}c^\top Mc}\right)\Bigg|_{c=c_1}^{c_2} = \int_\gamma f(u)e^{-\pi u^2}\,du. \tag{4.39}$$

We give an upper bound for

$$\max_{\lambda\in[0,1]}\left|e^{-\pi\gamma(\lambda)^2}e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right)\right| \tag{4.40}$$

$$= e^{\pi y^\top M^{-1}y}\max_{\lambda\in[0,1]} e^{\frac{-\pi}{-\frac{1}{2}c(\lambda)^\top Mc(\lambda)}\left(c(\lambda)^\top M\left(n+M^{-1}y\right)\right)^2}e^{-\pi\left(n+M^{-1}y\right)^\top M\left(n+M^{-1}y\right)} \tag{4.41}$$

$$= e^{\pi y^\top M^{-1}y}\max_{\lambda\in[0,1]} e^{-\pi\left(n+M^{-1}y\right)^\top A(\lambda)\left(n+M^{-1}y\right)} \tag{4.42}$$

$$\leq e^{\pi y^\top M^{-1}y}e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}. \tag{4.43}$$

Thus,

$$\left|\mathcal{E}_f\left(\frac{c^\top(Mn+y)}{-\frac{1}{2}c^\top Mc}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n+n^\top z\right)\right|$$

$$\leq \int_{\gamma_n}|f(u)|\,e^{\pi y^\top M^{-1}y}e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}\,du \tag{4.44}$$

$$\leq p(n)e^{-\pi\varepsilon\left\|n+M^{-1}y\right\|^2}, \tag{4.45}$$

where $\log p(n) = o\left(\|n\|^2\right)$. Thus, the terms of the series are $o\left(e^{-\frac{\pi\varepsilon}{2}\left(\|n\|^2+\|M^{-1}y\|\right)}\right)$, and so the series converges absolutely and uniformly for $x\in\mathbb{R}^g$ and $y\in K$. □

4.4. **Transformation laws of indefinite theta functions.** We will now prove the elliptic and modular transformation laws for indefinite theta functions. In all of these results, we assume that $z\in\mathbb{C}^g$, $\Omega\in\mathfrak{H}_g^{(1)}$, $c_j\in\mathbb{C}^g$ satisfying $\overline{c_j}^\top\operatorname{Im}(\Omega)c_j$, and f is a function of one variable satisfying the conditions specified in Definition 4.10.

Proposition 4.12. *The indefinite theta function attached to f satisfies the following transformation law with respect to the z variable, for $a+\Omega b\in\mathbb{Z}^g+\Omega\mathbb{Z}^g$:*

$$\Theta^{c_1,c_2}[f](z+a+\Omega b;\Omega) = e\left(-\frac{1}{2}b^\top\Omega b-b^\top z\right)\Theta^{c_1,c_2}[f](z;\Omega). \tag{4.46}$$

Proof. By definition,

$$\Theta^{c_1,c_2}[f](z+a+\Omega b;\Omega)$$

$$= \sum_{n\in\mathbb{Z}^g}\mathcal{E}_f\left(\frac{c^\top\operatorname{Im}(\Omega n+(z+a+\Omega b))}{-\frac{1}{2}c^\top\operatorname{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n)+n^\top(z+a+\Omega b)\right). \tag{4.47}$$

Because $a \in \mathbb{Z}^g$, $\operatorname{Im}(a)$ is zero and $e(n^\top a) = 1$, so

$$\begin{aligned}
&\Theta^{c_1,c_2}[f](z + a + \Omega b; \Omega) \\
&= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(\Omega(n+b)+z)}{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n) + n^\top(z+\Omega b)\right) \qquad (4.48)\\
&= e\left(-\frac{1}{2}b^\top\Omega b\right)\sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(\Omega(n+b)+z)}{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n+b) + n^\top z\right) \qquad (4.49)\\
&= e\left(-\frac{1}{2}b^\top\Omega b\right)\sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(\Omega n+z)}{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n) + (n-b)^\top z\right) \qquad (4.50)\\
&= e\left(-\frac{1}{2}b^\top\Omega b - b^\top z\right)\Theta[f]^{c_1,c_2}(z;\Omega). \qquad (4.51)
\end{aligned}$$

The identity is proved. □

Proposition 4.13. *The indefinite theta function satisfies the following condition with respect to the c variable:*

$$\Theta^{c_1,c_3}[f](z;\Omega) = \Theta^{c_1,c_2}[f](z;\Omega) + \Theta^{c_2,c_3}[f](z;\Omega) \qquad (4.52)$$

Proof. Add the series termwise. □

Theorem 4.14. *The indefinite theta function satisfies the following transformation laws with respect to the Ω variable, where $A \in \mathbf{GL}_g(\mathbb{Z})$, $B \in \mathbf{M}_g(\mathbb{Z})$, $B = B^\top$:*

1. $\Theta^{c_1,c_2}[f](z; A^\top\Omega A) = \Theta^{Ac_1,Ac_2}[f](A^{-\top}z;\Omega)$.
2. $\Theta^{c_1,c_2}[f](z;\Omega+2B) = \Theta^{c_1,c_2}[f](z;\Omega)$.
3. *In the case where $f(u) = \mathbb{1}(u) = 1$, we have*

$$\Theta^{c_1,c_2}(z;-\Omega^{-1}) = \frac{e^{\pi i z^\top\Omega z}}{\sqrt{\det(i\Omega^{-1})}}\Theta^{-\overline{\Omega}^{-1}c_1,-\overline{\Omega}^{-1}c_2}(\Omega z;\Omega). \qquad (4.53)$$

Proof. The proof of (1) is a direct calculation.

$$\begin{aligned}
&\Theta^{c_1,c_2}[f](z; A^\top\Omega A) \\
&= \sum_{n\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(A^\top\Omega A n+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top A^\top\Omega A n + n^\top z\right) \qquad (4.54)\\
&= \sum_{m\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{c^\top \operatorname{Im}(A^\top\Omega m+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}m^\top\Omega m + \left(A^{-1}m\right)^\top z\right) \qquad (4.55)
\end{aligned}$$

by the change of basis $m = An$, so

$$\begin{aligned}
&\Theta^{c_1,c_2}[f](z; A^\top\Omega A) \\
&= \sum_{m\in\mathbb{Z}^g} \mathcal{E}_f\left(\frac{(Ac)^\top \operatorname{Im}(\Omega m+A^{-\top}z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}m^\top\Omega m + m^\top A^{-\top} z\right) \qquad (4.56)\\
&= \Theta^{Ac_1,Ac_2}[f](A^{-\top}z;\Omega). \qquad (4.57)
\end{aligned}$$

The proof of (2) is also a direct calculation.

$$\begin{aligned}
&\Theta^{c_1,c_2}[f](z;\Omega+2B)\\
&=\sum_{n\in\mathbb{Z}^g}\mathcal{E}_f\left(\frac{c^\top \operatorname{Im}((\Omega+2B)n+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top(\Omega+2B)n+n^\top z\right) && (4.58)\\
&=\sum_{n\in\mathbb{Z}^g}\mathcal{E}_f\left(\frac{c^\top (\operatorname{Im}((\Omega)n+z))+2\operatorname{Im}(B)n}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n)+n^\top Bn+n^\top z\right) && (4.59)\\
&=\sum_{n\in\mathbb{Z}^g}\mathcal{E}_f\left(\frac{c^\top \operatorname{Im}((\Omega)n+z)}{\sqrt{-\frac{1}{2}c^\top \operatorname{Im}(\Omega)c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega(n)+n^\top z\right) && (4.60)\\
&=\Theta^{c_1,c_2}[f](z;\Omega); && (4.61)
\end{aligned}$$

where $e\left(n^\top Bn\right)=1$ because the $n^\top Bn$ are integers, and $\operatorname{Im}(B)=0$ because B is a real matrix.

The proof of (3) is more complicated, and, like the proof of the analogous property for definite (Jacobi and Riemann) theta functions, uses Poisson summation. The argument that follows is a modification of the argument that appears in the proof of Lemma 2.8 of Zwegers's thesis [20].

We will find a formula for the Fourier transform of the terms of our theta series. Most of the work is done in the calculation of the integral that follows. In this calculation, $M=\operatorname{Im}\Omega$, and $z=x+iy$ for $x,y\in\mathbb{C}^g$. The differential operator ∇_x is a row vector with entries $\frac{\partial}{\partial x_j}$, and similarly for ∇_n.

$$\begin{aligned}
&\nabla_x\left(\int_{n\in\mathbb{R}^g}\mathcal{E}\left(\frac{c^\top Mn+c^\top y}{\sqrt{-\frac{1}{2}c^\top Mc}}\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\,dn\right)\\
&=\int_{n\in\mathbb{R}^g}\mathcal{E}\left(\frac{c^\top Mn+c^\top y}{\sqrt{-\frac{1}{2}c^\top Mc}}\right)\Bigg|_{c=c_1}^{c_2} \nabla_x\left(e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\right)\,dn && (4.62)\\
&=\left(\int_{n\in\mathbb{R}^g}\mathcal{E}\left(\frac{c^\top Mn+c^\top y}{\sqrt{-\frac{1}{2}c^\top Mc}}\right)\Bigg|_{c=c_1}^{c_2} \nabla_n\left(e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\right)\,dn\right)\Omega^{-1} && (4.63)\\
&=\left(-\int_{n\in\mathbb{R}^g}\nabla_n\left(\mathcal{E}\left(\frac{c^\top Mn+c^\top y}{\sqrt{-\frac{1}{2}c^\top Mc}}\right)\right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\,dn\right)\Omega^{-1} && (4.64)\\
&=\left(k\int_{n\in\mathbb{R}^g}e\left(\frac{i}{-c^\top Mc}\left(c^\top \operatorname{Im}(\Omega)n\right)^2\right)e\left(Q_\Omega(n+a_z)\right)\,dn\right)c^\top M\Omega^{-1}\Bigg|_{c=c_1}^{c_2}, && (4.65)
\end{aligned}$$

where $k = \frac{-2}{\sqrt{-\frac{1}{2}c^\top Mc}} \in \mathbb{C}$, $a_z = \Omega^{-1}z - M^{-1}y \in \mathbb{C}^g$, and integration by parts was used in eq. (4.64). Continuing the calculation,

$$\nabla_x \left(\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top Mn + c^\top y}{\sqrt{-\frac{1}{2}c^\top Mc}} \right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\, dn \right)$$

$$= k\left(\int_{n\in\mathbb{R}^g} e\left(Q_{\Omega - \frac{2i}{c^\top Mc}Mcc^\top M}(n) + a^\top \Omega n + \frac{1}{2}a^\top \Omega a \right) dn \right) c^\top M\Omega^{-1}\Bigg|_{c=c_1}^{c_2} \tag{4.66}$$

$$= ke\left(-\frac{1}{2}a^\top\Omega\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1}\Omega a + \frac{1}{2}a^\top\Omega a \right) I^{(c)}c^\top M\Omega^{-1}\Bigg|_{c=c_1}^{c_2}, \tag{4.67}$$

where

$$I^{(c)} = \int_{n\in\mathbb{R}^g} e\left(Q_{\Omega - \frac{2i}{c^\top Mc}Mcc^\top M}\left(n + \left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1}\Omega a \right) \right) dn \tag{4.68}$$

$$= \frac{1}{\det\sqrt{-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)}} \tag{4.69}$$

by Lemma 3.4.

We can check (by multiplication) that

$$\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1} = \Omega^{-1} - \frac{2i}{c^\top Mc - 2ic^\top M\Omega^{-1}Mc}\Omega^{-1}Mcc^\top M\Omega^{-1}. \tag{4.70}$$

Thus,

$$\Omega - \Omega\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)^{-1}\Omega = \frac{2i}{c^\top Mc - 2ic^\top M\Omega^{-1}Mc}Mcc^\top M. \tag{4.71}$$

Now compute, using Lemma 4.7, $Ma = M\Omega^{-1}z - y = \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)z - y = \overline{\Omega}\left(\mathrm{Im}\left(-\Omega^{-1}\right)z - \overline{\Omega}^{-1}y\right) = \frac{1}{2i}\overline{\Omega}\left(\left(-\Omega^{-1} + \overline{\Omega}^{-1}\right)z - \overline{\Omega}^{-1}(z - \overline{z})\right) = \frac{1}{2i}\overline{\Omega}\left(-\Omega^{-1}z + \overline{\Omega}^{-1}\overline{z}\right) = \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}z\right)$. Also by Lemma 4.7, $M - 2iM\Omega^{-1}M = \overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}$, and

$$\sqrt{\det\left(-i\left(\Omega - \frac{2i}{c^\top Mc}Mcc^\top M\right)\right)} = \sqrt{\det(-i\Omega)}\frac{\sqrt{-c^\top\overline{\Omega}\,\mathrm{Im}\left(-\Omega^{-1}\right)\overline{\Omega}c}}{\sqrt{-c^\top Mc}}. \tag{4.72}$$

We have now shown that

$$\nabla_x \left(\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}\left(\Omega n + z\right)}{\sqrt{-\frac{1}{2}c^\top M c}} \right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\, dn \right)$$

$$= \frac{-2e\left(\frac{i}{(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}(c^\top M a)^2\right)}{\sqrt{\det(-i\Omega)}\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}} c^\top M \Omega^{-1}\Bigg|_{c=c_1}^{c_2} \tag{4.73}$$

$$= \frac{-2e\left(\frac{i}{(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}(c^\top M a)^2\right)}{\sqrt{\det(-i\Omega)}\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}} (\Omega c)^\top \operatorname{Im}(\Omega^{-1})\Bigg|_{c=c_1}^{c_2} \tag{4.74}$$

$$= \frac{1}{\sqrt{\det(-i\Omega)}} \nabla_x \mathcal{E}\left(\frac{(\overline{\Omega}c)^\top\left(\operatorname{Im}(-\Omega^{-1})n + \operatorname{Im}(-\Omega^{-1}z)\right)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}} \right)\Bigg|_{c=c_1}^{c_2}. \tag{4.75}$$

Define the following function on $\mathbb{C}^g$,

$$C(z) := \int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}\left(\Omega n + z\right)}{\sqrt{-\frac{1}{2}c^\top \Omega c}} \right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\, dn \tag{4.76}$$

$$- \frac{1}{\sqrt{\det(i\Omega)}} \mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}} \right)\Bigg|_{c=c_1}^{c_2}, \tag{4.77}$$

suppressing the dependence of $C(z)$ on Ω and c. We have just shown that $\nabla_x C(z) = 0$, so $C(z+a) = C(z)$ for any $a \in \mathbb{R}^g$. By inspection, $C(z+\Omega^{-1}b) = C(z)$ for any $b \in \mathbb{R}^g$. It follow from both of these properties that $C(z)$ is constant. Moreover, by inspection, $C(-z) = -C(z)$; therefore, $C(z) = 0$. In other words,

$$\int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}\left(\Omega n + z\right)}{\sqrt{-\frac{1}{2}c^\top \Omega c}} \right)\Bigg|_{c=c_1}^{c_2} e\left(Q_\Omega\left(n+\Omega^{-1}z\right)\right)\, dn$$

$$= \frac{1}{\sqrt{\det(-i\Omega)}} \mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}} \right)\Bigg|_{c=c_1}^{c_2}. \tag{4.78}$$

Now set $g(z) := \Theta^{c_1,c_2}(z;\Omega)$, which has Fourier coefficients

$$c_n(g)(z) = \mathcal{E}\left(\frac{c^\top \operatorname{Im}\left(\Omega n + z\right)}{\sqrt{-\frac{1}{2}c^\top \Omega c}} \right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right). \tag{4.79}$$

By plugging in $z-\nu$ for z in eq. (4.78) and multiplying both sides by $e\left(-\frac{1}{2}(z-\nu)^\top\Omega^{-1}(z-\nu)\right)$, we obtain the following expression for the Fourier coefficients of $\hat{g}$:

$$c_\nu\left(\hat{g}\right)(z) = \int_{n\in\mathbb{R}^g} \mathcal{E}\left(\frac{c^\top \operatorname{Im}\left(\Omega n + z\right)}{\sqrt{-\frac{1}{2}c^\top\Omega c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top\Omega n + n^\top z\right) e(-n^\top\nu)\,dn \tag{4.80}$$

$$= \frac{e\left(-\frac{1}{2}(z-\nu)^\top\Omega^{-1}(z-\nu)\right)}{\sqrt{\det(-i\Omega)}}\,\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}\nu-\Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right)\Bigg|_{c=c_1}^{c_2} \tag{4.81}$$

$$= \frac{e\left(-\frac{1}{2}z^\top\Omega^{-1}z\right)}{\sqrt{\det(-i\Omega)}}\,\mathcal{E}\left(\frac{(\overline{\Omega}c)^\top \operatorname{Im}(-\Omega^{-1}(-\nu)-\Omega^{-1}z)}{\sqrt{-\frac{1}{2}(\overline{\Omega}c)\operatorname{Im}(-\Omega^{-1})(\overline{\Omega}c)}}\right)\Bigg|_{c=c_1}^{c_2} \tag{4.82}$$

$$\cdot\, e\left(\frac{1}{2}\nu^\top(-\Omega^{-1})\nu + (-\nu)^\top(-\Omega^{-1}z)\right). \tag{4.83}$$

It follows by Poisson summation that

$$\Theta^{c_1,c_2}(z;\Omega) = \sum_{\nu\in\mathbb{Z}^g} c_\nu\left(\hat{g}\right)(z) \tag{4.84}$$

$$= \frac{e\left(-\frac{1}{2}z^\top\Omega^{-1}z\right)}{\sqrt{\det(-i\Omega)}}\Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}z;-\Omega^{-1}\right). \tag{4.85}$$

We obtain (3) by replacing Ω with $-\Omega^{-1}$. □

4.5. **Indefinite theta functions with characteristics.** Now we restate the transformation laws using "characteristics" notation, which will be used when we define indefinite zeta functions in section 6.

Definition 4.15. Define the *indefinite theta null with characteristics* $p,q\in\mathbb{R}^g$:

$$\Theta^{c_1,c_2}_{p,q}[f](\Omega) = e^{2\pi i\left(\frac{1}{2}q^\top\Omega q + p^\top q\right)}\Theta^{c_1,c_2}[f]\left(p+\Omega q;\Omega\right); \tag{4.86}$$

$$\Theta^{c_1,c_2}_{p,q}(\Omega) = e^{2\pi i\left(\frac{1}{2}q^\top\Omega q + p^\top q\right)}\Theta^{c_1,c_2}\left(p+\Omega q;\Omega\right). \tag{4.87}$$

The transformation laws for $\Theta^{c_1,c_2}_{p,q}[f](\Omega)$ follow from the transformation laws for $\Theta^{c_1,c_2}[f](z;\Omega)$.

Proposition 4.16. *The elliptic transformation law for the indefinite theta null with characteristics is:*

$$\Theta^{c_1,c_2}_{p+a,q+b}[f](\Omega) = e(a^\top(q+b))\Theta^{c_1,c_2}_{p,q}[f](\Omega). \tag{4.88}$$

Proposition 4.17. *The modular transformation laws for the indefinite theta null with characteristics are as follows.*

(1) $\Theta^{c_1,c_2}_{p,q}[f](A^\top\Omega A) = \Theta^{Ac_1,Ac_2}_{A^{-\top}p,Aq}[f](\Omega)$.
(2) $\Theta^{c_1,c_2}_{p,q}[f](\Omega+2B) = e(-q^\top Bq)\Theta^{c_1,c_2}_{p+2Bq,q}[f](\Omega)$.
(3) $\Theta^{c_1,c_2}_{p,q}(-\Omega^{-1}) = \frac{e(p^\top q)}{\sqrt{\det(i\Omega^{-1})}}\Theta^{-\overline{\Omega}^{-1}c_1,-\overline{\Omega}^{-1}c_2}_{-q,p}(\Omega)$.

4.6. **P-stable indefinite theta functions.** We now introduce a special symmetry that may be enjoyed by the parameters (c_1, c_2, z, Ω), which we call *P-stability*. In this section, c_1, c_2 will always be real vectors.

Definition 4.18. Let $P \in \mathbf{GL}_g(\mathbb{Z})$ be fixed. Let $z \in \mathbb{C}^g$, $\Omega \in \mathfrak{H}_g^{(1)}$, $c_1, c_2 \in \mathbb{R}^g$ satisfying $c_j^\top \operatorname{Im}(\Omega) c_j < 0$. The quadruple (c_1, c_2, z, Ω) is called *P-stable* if $P^\top \Omega P = \Omega$, $Pc_1 = c_2$, and $P^\top z \equiv z \pmod{\mathbb{Z}^2}$.

Remarkably, P-stable indefinite theta functions attached to $f(u) = |u|^r$ turn out to be independent of r (up to a constant factor).

Theorem 4.19 (P-Stability Theorem)**.** *Set $\Theta_r^{c_1,c_2}(z;\Omega) := \frac{\pi^{\frac{r+1}{2}}}{\Gamma\left(\frac{r+1}{2}\right)}\Theta^{c_1,c_2}[f](z;\Omega)$ when $f(u) = |u|^r$ for $\operatorname{Re}(r) > -1$. If (c_1, c_2, z, Ω) is P-stable, then $\Theta_r^{c_1,c_2}(z,\Omega)$ is independent of r.*

Proof. Let $M = \operatorname{Im}(\Omega)$ and $y = \operatorname{Im}(z)$. If $\alpha \in \mathbb{R}$ and $\operatorname{Re}(r) > 1$, then

$$\mathcal{E}_r(\alpha) = \int_0^\alpha |u|^r e^{-\pi u^2}\, du \tag{4.89}$$

$$= \operatorname{sgn}(\alpha) \int_0^{|\alpha|} u^r e^{-\pi u^2}\, du \tag{4.90}$$

$$= -\frac{\operatorname{sgn}(\alpha)}{2\pi} \int_0^{|\alpha|} u^{r-1}\, d\left(e^{-\pi u^2}\right) \tag{4.91}$$

$$= -\frac{\operatorname{sgn}(\alpha)}{2\pi} \left(u^{r-1} e^{-\pi u^2}\Big|_{u=0}^{|\alpha|} - \int_0^{|\alpha|} e^{-\pi u^2}\, d\left(u^{r-1}\right) \right) \tag{4.92}$$

$$= -\frac{\operatorname{sgn}(\alpha)}{2\pi} \left(|\alpha|^{r-1} e^{-\pi\alpha^2} - (r-1)\int_0^{|\alpha|} u^{r-2} e^{-\pi u^2}\, du \right) \tag{4.93}$$

$$= \frac{1}{2\pi} \left(-\operatorname{sgn}(\alpha)\, |\alpha|^{r-1} e^{-\pi\alpha^2} + (r-1)\mathcal{E}_{r-2}(\alpha) \right). \tag{4.94}$$

Let $\alpha_n^c = \frac{c^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-Q_M(c)}}$. Set $A^c := M + M \operatorname{Re}\left((-Q_M(c))^{-1} cc^\top\right) M$, so that A^{c_1} and A^{c_2} are positive definite, as in the proof of Proposition 4.11. Thus,

$$\Theta_r^{c_1,c_2}(z,\Omega) = -\frac{\pi^{r/2}}{\Gamma\left(\frac{r+1}{2}\right)} S + \Theta_{r-2}^{c_1,c_2}(z,\Omega), \tag{4.95}$$

where

$$S = \sum_{n \in \mathbb{Z}^g} \operatorname{sgn}\left(\alpha_n^c\right) \left|\alpha_n^c\right|^{r-1} \exp\left(-\pi \left(\alpha_n^c\right)^2\right)\Big|_{c=c_1}^{c_2} e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right). \tag{4.96}$$

The c_1 and c_2 terms in this sum decay exponentially, because

$$\left|\exp\left(-\pi \left(\alpha_n^c\right)^2\right) e\left(\frac{1}{2} n^\top \Omega n + n^\top z\right)\right| = \exp\left(-2\pi Q_{A^c}\left(n + M^{-1} y\right)\right). \tag{4.97}$$

Thus, the series may be split as a sum of two series:

$$S = \sum_{n\in\mathbb{Z}^g} \operatorname{sgn}\left(\alpha_n^{c_2}\right) \left|\alpha_n^{c_2}\right|^{r-1} \exp\left(-\pi \left(\alpha_n^{c_2}\right)^2\right) e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right) \tag{4.98}$$

$$- \sum_{n\in\mathbb{Z}^g} \operatorname{sgn}\left(\alpha_n^{c_1}\right) \left|\alpha_n^{c_1}\right|^{r-1} \exp\left(-\pi \left(\alpha_n^{c_1}\right)^2\right) e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right). \tag{4.99}$$

Now we use the P-symmetry to show that these two series are, in fact, equal. Note that $\operatorname{Im}(P^\top z) = \operatorname{Im}(z)$ because $P^\top z \equiv z \pmod{\mathbb{Z}^2}$, so

$$\alpha_{Pn}(c_2) = \frac{(Pc_1)^\top \operatorname{Im}(\Omega P n + z)}{\sqrt{-Q_M(Pc_1)}} \tag{4.100}$$

$$= \frac{c_1^\top \operatorname{Im}(P^\top \Omega P n + P^\top z)}{\sqrt{-Q_{P^\top M P}(c_1)}} \tag{4.101}$$

$$= \frac{c_1^\top \operatorname{Im}(\Omega n + z)}{\sqrt{-Q_M(c_1)}} \tag{4.102}$$

$$= \alpha_n(c_1). \tag{4.103}$$

Moreover,

$$\frac{1}{2}(Pn)^\top \Omega (Pn) + (Pn)^\top z = \frac{1}{2}n^\top (P^\top \Omega P) n + n^\top (P^\top z) \tag{4.104}$$

$$\equiv \frac{1}{2}n^\top \Omega n + n^\top z \pmod{\mathbb{Z}^2}. \tag{4.105}$$

Thus, we may substitute Pn for n in the first series (involving c_2) to obtain the second (involving c_1).

We've now shown the periodicity relation

$$\Theta_r^{c_1,c_2}(z,\Omega) = \Theta_{r-2}^{c_1,c_2}(z,\Omega). \tag{4.106}$$

Note that this identity provides an analytic continuation of $\Theta_r^{c_1,c_2}(z,\Omega)$ to the entire r-plane. To show that it is constant in r, we will show that it is bounded on vertical strips in the r-plane. As in the proof of Proposition 4.11, bound $(x,\lambda) \mapsto x^\top A(\lambda) x$, considered as a positive real-valued continuous function on the product of the unit ball $\{x^\top x = 1\}$ and the interval $[0,1]$, from below by its global minimum $\varepsilon > 0$. Thus,

$$\left| \mathcal{E}_r\left(\frac{c^\top(Mn+y)}{\sqrt{-\frac{1}{2}c^\top M c}}\right)\Bigg|_{c=c_1}^{c_2} e\left(\frac{1}{2}n^\top \Omega n + n^\top z\right)\right|$$

$$\le \left| \int_{\frac{c_1^\top \operatorname{Im}(\Omega n+z)}{\sqrt{-\frac{1}{2}c_1^\top \operatorname{Im}(\Omega) c_1}}}^{\frac{c_2^\top \operatorname{Im}(\Omega n+z)}{\sqrt{-\frac{1}{2}c_2^\top \operatorname{Im}(\Omega) c_2}}} |u|^{\operatorname{Re}(r)}\, du \right| e^{\pi y^\top M^{-1} y} e^{-\pi\varepsilon \left\|n + M^{-1}y\right\|^2} \tag{4.107}$$

$$\le p_{\operatorname{Re}(r)}(n) e^{-\pi\varepsilon \left\|n+M^{-1}y\right\|^2}, \tag{4.108}$$

where $p_{\operatorname{Re}(r)}(n)$ is a polynomial independent of $\operatorname{Im}(r)$. Hence, $\Theta_r^{c_1,c_2}(z,\Omega)$ is bounded on the line $\operatorname{Re}(r) = \sigma$ by $\sum_{n\in\mathbb{Z}^g} p_\sigma(n) e^{-\pi\varepsilon \left\|n+M^{-1}y\right\|^2}$. It follows that it is bounded on any vertical strip. Along with periodicity, this implies that $\Theta_r^{c_1,c_2}(z,\Omega)$ as a function of r is bounded and entire, thus constant. □

5. Definite zeta functions and real analytic Eisenstein series

We will now consider the Mellin transforms of definite and indefinite theta functions. In the definite case in dimension 2, they are generalize analytic Eisenstein series, and they specialize to zeta functions of imaginary quadratic ideal classes. In the indefinite case in dimension 2, we recover certain L-series attached to (ideals of orders of) real quadratic fields. The class of L-series we recover spans the same vector space as those Hecke L-functions attached to Hecke characters of finite order ramified at exactly one infinite place.

We define the definite zeta function as a Mellin transform of the indefinite theta null with real characteristics.

Definition 5.1. Let $\Omega \in \mathfrak{H}_g^{(0)}$ and $p, q \in \mathbb{R}^g$. The *definite zeta function* is

$$\hat{\zeta}_{p,q}(\Omega, s) = \begin{cases} \int_0^\infty \Theta_{p,q}(t\Omega) t^s \frac{dt}{t} & \text{if } q \notin \mathbb{Z}^g, \\ \int_0^\infty \left(\Theta_{p,q}(t\Omega) - 1\right) t^s \frac{dt}{t} & \text{if } q \in \mathbb{Z}^g. \end{cases} \tag{5.1}$$

By direct calculation, $\hat{\zeta}_{p,q}(\Omega, s)$ has a Dirichlet series expansion.

$$\hat{\zeta}_{p,q}(\Omega, s) = (2\pi)^{-s}\Gamma(s) \sum_{\substack{n \in \mathbb{Z}^g \\ n \neq -q}} e(p^\top (n+q)) Q_{-i\Omega}(n+q)^{-s}, \tag{5.2}$$

where $Q_{-i\Omega}(n+q)^{-s}$ is defined using the standard branch of the logarithm (with a branch cut on the negative real axis).

Now, suppose $g = 2$, $\Omega = iM$ for some real symmetric, positive definite matrix M, $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $q \notin \mathbb{Z}^2$. Then the definite zeta function may be written as follows.

$$\hat{\zeta}_{0,q}(\Omega, s) = (2\pi)^{-s}\Gamma(s) \sum_{n \in \mathbb{Z}^2} Q_M(n+q)^{-s} \tag{5.3}$$

$$= (2\pi)^{-s}\Gamma(s) \sum_{n \in \mathbb{Z}^2 + q} Q_M(n)^{-s}. \tag{5.4}$$

Up to scaling, M is of the form $M = \frac{1}{\operatorname{Im}(\tau)} \begin{pmatrix} 1 & \operatorname{Re}(\tau) \\ \operatorname{Re}(\tau) & \tau\overline{\tau} \end{pmatrix}$ for some $\tau \in \mathfrak{H}$; scaling M by $\lambda \in \mathbb{R}$ simply scales $\hat{\zeta}_{p,q}(\Omega, s)$ by λ^{-s}, so we assume M is of this form. Write

$$Q_M \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \frac{1}{2\operatorname{Im}(\tau)} \left(n_1^2 + 2\operatorname{Re}\tau n_1 n_2 + \tau\overline{\tau} n_2^2\right) \tag{5.5}$$

$$= \frac{1}{2\operatorname{Im}(\tau)} \left|n_1 + n_2\tau\right|^2 \tag{5.6}$$

Thus,

$$\hat{\zeta}_{0,q}(\Omega, s) = \pi^{-s}\Gamma(s) \operatorname{Im}(\tau)^s \sum_{\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{Z}^2 + q} \left|n_1\tau + n_2\right|^{-2s}. \tag{5.7}$$

If $q \in \mathbb{Q}^2$ and the gcd of the denominators of the entries of q is N, this is essentially an Eisenstein series of associated to $\Gamma_1(N)$. Choose $k, \ell \in \mathbb{Z}$ such that $q \equiv \begin{pmatrix} k/N \\ \ell/N \end{pmatrix} \pmod 1$ and $\gcd(k, \ell) = 1$. Then, we have

$$\hat{\zeta}_{0,q}(\Omega, s) = (\pi N)^{-s}\Gamma(s)\operatorname{Im}(\tau)^s \sum_{\substack{c \equiv k \pmod N \\ d \equiv \ell \pmod N}} |c\tau + d|^{-2s}. \tag{5.8}$$

The Eisenstein series associated to the cusp ∞ of $\Gamma_1(N)$ is

$$E^{\infty}_{\Gamma_1(N)}(\tau, s) = \sum_{\gamma \in \Gamma_1^{\infty}(N)\backslash\Gamma_1(N)} \operatorname{Im}(\gamma \cdot \tau)^s \tag{5.9}$$

$$= \operatorname{Im}(\tau)^s \sum_{\substack{c \equiv 0 \pmod N \\ d \equiv 1 \pmod N}} |c\tau + d|^{-2s}. \tag{5.10}$$

Here, $\Gamma_1^{\infty}(N)$ is the stabilizer of ∞ under the fractional linear transformation action; that is, $\Gamma_1^{\infty}(N) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$.

Choose $u, v \in \mathbb{Z}$ such that $\det \begin{pmatrix} u & v \\ k & \ell \end{pmatrix} = 1$. We have

$$E^{\infty}_{\Gamma_1(N)}\left(\frac{u\tau + v}{k\tau + \ell}, s\right) = \operatorname{Im}\left(\frac{u\tau + v}{k\tau + \ell}\right)^s \sum_{\substack{c \equiv 0 \pmod N \\ d \equiv 1 \pmod N}} \left| c\left(\frac{u\tau + v}{k\tau + \ell}\right) + d \right|^{-2s} \tag{5.11}$$

$$= \operatorname{Im}(\tau)^s \sum_{\substack{c \equiv 0 \pmod N \\ d \equiv 1 \pmod N}} |(cu + dk)\tau + (cv + d\ell)|^{-2s} \tag{5.12}$$

$$= \operatorname{Im}(\tau)^s \sum_{\substack{c' \equiv k \pmod N \\ d' \equiv \ell \pmod N}} |c'\tau + d'|^{-2s}. \tag{5.13}$$

Combining eq. (5.8) and eq. (5.13), we see that

$$\hat{\zeta}_{0,q}(\Omega, s) = (\pi N)^{-s}\Gamma(s) E^{\infty}_{\Gamma_1(N)}\left(\frac{u\tau + v}{k\tau + \ell}, s\right). \tag{5.14}$$

6. Indefinite zeta functions: definition, analytic continuation, and functional equation

As usual, let $\Omega \in \mathfrak{H}_g^{(1)}$, $p, q \in \mathbb{R}^g$, $c_1, c_2 \in \mathbb{C}^g$, $\overline{c_1}^\top M c_1 < 0$, $\overline{c_2}^\top M c_2 < 0$.

We define the indefinite zeta function using a Mellin transform of the indefinite theta function with characteristics.

Definition 6.1. The indefinite zeta function is

$$\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = \int_0^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega) t^s \frac{dt}{t}. \tag{6.1}$$

The terminology "zeta function" here should not be taken to mean that $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s)$ has a Dirichlet series—it (usually) doesn't (although it does have an analogous series expansion involving hypergeometric functions, as we'll see in section 7). Rather, we think of it as a zeta function by analogy with the definite case, and (as we'll see) because is sometimes specializes to certain classical zeta functions.

By defining the zeta function as a Mellin transform, we've set things up so that a proof of the functional equation Theorem 1.1 is a natural first step. Analytic continuation and a functional equation will follow from Theorem 4.14 by standard techniques. Our analytic continuation also gives an expression that converges quickly everywhere and is therefore useful for numerical computation, unlike eq. (6.1) or the series expansion in section 7.

Theorem 1.1. The function $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s)$ may be analytically continued to an entire function on $\mathbb{C}$. It satisfies the functional equation

$$\hat{\zeta}_{p,q}^{c_1,c_2}\left(\Omega, \frac{g}{2} - s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \hat{\zeta}_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}\left(-\Omega^{-1}, s\right). \tag{6.2}$$

Proof. Fix $r > 0$, and split up the Mellin transform integral into two pieces,

$$\begin{aligned}\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) &= \int_0^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega)t^s \frac{dt}{t} &(6.3)\\ &= \int_r^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega)t^s \frac{dt}{t} + \int_0^r \Theta_{p,q}^{c_1,c_2}(t\Omega)t^s \frac{dt}{t}. &(6.4)\end{aligned}$$

Replacing t by t^{-1}, and then using part (3) of Theorem 4.14, the second integral is

$$\begin{aligned}\int_0^r \Theta_{p,q}^{c_1,c_2}(t\Omega)t^s \frac{dt}{t} &= \int_{r^{-1}}^\infty \Theta_{p,q}^{c_1,c_2}(t^{-1}\Omega)t^{-s} \frac{dt}{t} &(6.5)\\ &= \int_{r^{-1}}^\infty \frac{e(p^\top q)}{\sqrt{\det(-it\Omega)}} \Theta_{-q,p}^{t\overline{\Omega}c_1,t\overline{\Omega}c_2}(-(t^{-1}\Omega)^{-1})t^{-s} \frac{dt}{t} &(6.6)\\ &= \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^\infty \Theta_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(t(-\Omega^{-1}))t^{\frac{g}{2}-s} \frac{dt}{t}. &(6.7)\end{aligned}$$

(Recall that scaling the c_j does not affect the value of $\Theta_{p,q}^{c_1,c_2}(\Omega)$.) Putting it all together, we have

$$\begin{aligned}\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = &\int_r^\infty \Theta_{p,q}^{c_1,c_2}(t\Omega)t^s \frac{dt}{t}\\ &+ \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}} \int_{r^{-1}}^\infty \Theta_{-q,p}^{\overline{\Omega}c_1,\overline{\Omega}c_2}(t(-\Omega^{-1}))t^{\frac{g}{2}-s} \frac{dt}{t}. &(6.8)\end{aligned}$$

As we showed in the proof of Proposition 4.11, the Θ-functions in both integrals decay exponentially as $t \to \infty$, so the right-hand side converges for all $s \in \mathbb{C}$. The right-hand side is obviously analytic for all $s \in \mathbb{C}$, so we've analytically continued $\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s)$ to an entire function of s. Finally, we must prove the functional equation. If we plug $\frac{g}{2} - s$ for s in eq. (6.8), factor out the coefficient of the second term, and switch the order of the two terms, we obtain

$$\hat{\zeta}^{c_1,c_2}_{p,q}\left(\Omega, \frac{g}{2}-s\right) = \frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}}\left(\int_{r^{-1}}^{\infty} \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(t(-\Omega^{-1}))t^s\frac{dt}{t}\right.$$
$$\left. -\frac{e(-p^\top q)}{\sqrt{\det(i\Omega^{-1})}}\int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega)t^{\frac{g}{2}-s}\frac{dt}{t}\right). \tag{6.9}$$

Reusing eq. (6.8) on $\hat{\zeta}^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}\left(-\Omega^{-1}, s\right)$, and appealing to the fact that $\Theta^{c_1,c_2}_{p,q}(\Omega) = -\Theta^{c_1,c_2}_{-p,-q}(\Omega)$, we have

$$\hat{\zeta}^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}\left(-\Omega^{-1}, s\right) = \int_{r^{-1}}^{\infty} \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(t(-\Omega^{-1}))t^s\frac{dt}{t}$$
$$-\frac{e(-p^\top q)}{\sqrt{\det(i\Omega^{-1})}}\int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega)t^{\frac{g}{2}-s}\frac{dt}{t}. \tag{6.10}$$

The functional equation now follows from eq. (6.9) and eq. (6.10). □

The formula for the analytic continuation is useful in itself. In particular, we have used this formula for computer calculations, as it may be used to compute the indefinite zeta function to arbitrary precision in polynomial time.

Corollary 6.2. *The following expression is valid on the entire s-plane.*

$$\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s) = \int_r^\infty \Theta^{c_1,c_2}_{p,q}(t\Omega)t^s\frac{dt}{t}$$
$$+\frac{e(p^\top q)}{\sqrt{\det(-i\Omega)}}\int_{r^{-1}}^{\infty} \Theta^{\overline{\Omega}c_1,\overline{\Omega}c_2}_{-q,p}(t(-\Omega^{-1}))t^{\frac{g}{2}-s}\frac{dt}{t}. \tag{6.11}$$

Proof. The is eq. (6.8). □

7. Series expansion of indefinite zeta function

In this section, we give a series expansion for indefinite zeta functions, under the assumption that c_1 and c_2 are real. Specifically, we write $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega, s)$ as a sum of three series, the first of which is a Dirichlet series and the others of which involve hypergeometric functions. This expansion is related to the decomposition of a weak harmonic Maass form into its holomorphic "mock" piece and a nonholomorphic piece obtained from a "shadow" form in another weight. However, we don't describe the relationship here.

To proceed, we will need to introduce some special functions and review some of their properties.

7.1. **Hypergeometric functions and modified beta functions.**

Let a, b, c be complex numbers, c not a negative integer or zero. If $z \in \mathbb{C}$ with $|z| < 1$, the power series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{z^n}{n!} \tag{7.1}$$

converges. Here we are using the Pochhammer symbol $(w)_n := w(w+1)\cdots(w+n-1)$.

Proposition 7.1. *There is an identity*

$$ {}_2F_1(a,b;c;z) = (1-z)^{-b} {}_2F_1\left(b, c-a; c; \frac{z}{z-1}\right), \tag{7.2}$$

valid about $z = 0$ and using the principal branch for $(1-z)^{-b}$.

Proof. This is part of Theorem 2.2.5 of [2]. □

Using this identity, we extend the domain of definition of ${}_2F_1(a,b;c;x)$ from the unit disc $\{|z| < 1\}$ to the union of the unit disc and a half-plane $\{|z| < 1\} \cup \{\mathrm{Re}(z) < \frac{1}{2}\}$. We interpret $(1-z)^{-b} = \exp(-b\log(1-z))$ with the logarithm having a branch cut along the negative real axis. At the boundary point $z = 1$, the hypergeometric series converges when $\mathrm{Re}(c) > \mathrm{Re}(a+b)$, and its evaluation is a classical theorem of Gauss.

Proposition 7.2. *If* $\mathrm{Re}(c) > \mathrm{Re}(a+b)$*, then*

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{7.3}$$

Proof. This is Theorem 2.2.2 of [2]. □

Of particular interest to us will be a special hypergeometric function which is a modified version of the beta function.

Definition 7.3. Let $x > 0$ and $a, b \in \mathbb{C}$. The *beta function* is

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt, \tag{7.4}$$

and the *modified beta function* is

$$\tilde{B}(x; a, b) = \int_0^x t^{a-1}(1+t)^{b-1}\,dt. \tag{7.5}$$

The following proposition enumerates some properties of the modified beta function.

Proposition 7.4. *Let $x > 0$, and let a, b be complex numbers with* $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ *and* $\mathrm{Re}(a+b) < 1$*. Then,*

(1) $\tilde{B}(x; a, b) = B\left(\frac{x}{x+1}; a, 1-a-b\right)$,
(2) $\tilde{B}(x; a, b) = \frac{1}{a} x^a {}_2F_1(a, 1-b; a+1; -x)$,
(3) $\tilde{B}\left(\frac{1}{x}; a, b\right) = \frac{\Gamma(a)\Gamma(1-a-b)}{\Gamma(1-b)} - \tilde{B}(x; 1-a-b, b)$, *and*
(4) $\tilde{B}(+\infty; a, b) = B(1; a, 1-a-b) = \frac{\Gamma(a)\Gamma(1-a-b)}{\Gamma(1-b)}$.

Proof. To prove (1), we use the substitution $t = \frac{u}{1-u}$.

$$\tilde{B}(x;a,b) = \int_0^x t^{a-1}(1+t)^{b-1}\,dt \tag{7.6}$$
$$= \int_0^{\frac{x}{x+1}} \left(\frac{u}{1-u}\right)^{a-1} \left(1+\frac{u}{1-u}\right)^{b-1} \frac{du}{(1-u)^2} \tag{7.7}$$
$$= \int_0^{\frac{x}{x+1}} u^{a-1}(1-u)^{-a-b}\,du \tag{7.8}$$
$$= B\left(\frac{x}{x+1}; a, 1-a-b\right). \tag{7.9}$$

To prove (2), expand $G(x;a,b)$ as a power series in x (up to a non-integral power).

$$\tilde{B}(x;a,b) = \int_0^x t^{a-1}(1+t)^{b-1}\,dt \tag{7.10}$$
$$= \int_0^x \sum_{n=0}^{\infty} \binom{b-1}{n} t^{n+a-1}\,dt \tag{7.11}$$
$$= \sum_{n=0}^{\infty} \binom{b-1}{n} \frac{1}{n+a} x^{n+a} \tag{7.12}$$
$$= \sum_{n=0}^{\infty} \frac{(b-n)\cdot(b-n+1)\cdots(b-1)}{n!} \cdot \frac{1}{n+a} x^{n+a} \tag{7.13}$$
$$= x^a \sum_{n=0}^{\infty} \frac{(-1)^n(1-b)\cdot(2-b)\cdots(n-b)}{n+a} \cdot \frac{x^n}{n!} \tag{7.14}$$
$$= x^a \sum_{n=0}^{\infty} \frac{(a)_n(1-b)_n}{a(a+1)_n} \cdot \frac{(-x)^n}{n!} \tag{7.15}$$
$$= \frac{1}{a} x^a {}_2F_1(a, 1-b; a+1; -x). \tag{7.16}$$

To prove (3), use the substitution $t = \frac{1}{u}$.

$$\tilde{B}\left(\frac{1}{x}; a, b\right) = \int_0^{1/x} t^{a-1}(1+t)^{b-1}\,dt \tag{7.17}$$
$$= \int_\infty^x u^{-a+1}\left(1+\frac{1}{u}\right)^{b-1} \left(-\frac{du}{u^2}\right) \tag{7.18}$$
$$= \int_x^\infty u^{-a-b}(1+u)^{b-1}\,du \tag{7.19}$$
$$= G(+\infty, 1-a-b, b) - G(x, 1-a-b, b) \tag{7.20}$$

To complete the proof of (3), we need to prove (4). Note that it follows from (4) that $\tilde{B}(+\infty, 1-a-b, b) = \frac{\Gamma(a)\Gamma(1-a-b)}{\Gamma(1-b)}$. The first equality of (4) follows from (1) with $x \to +\infty$;

we will now derive the second. By (2),

$$\begin{aligned}\tilde{B}(x;a,b) &= \frac{1}{a}x^a{}_2F_1(a,1-b;a+1;-x) && (7.21)\\ &= \frac{1}{a}x^a{}_2F_1(1-b,a;a+1;-x) && (7.22)\\ &= \frac{1}{a}x^a\cdot(1-(-x))^{-a}{}_2F_1\left(a,(a+1)-(1-b);a+1;\frac{-x}{(-x)-1}\right) && (7.23)\\ &= \frac{1}{a}\left(\frac{x}{1+x}\right)^a{}_2F_1\left(a,a+b;a+1;\frac{x}{x+1}\right) && (7.24)\end{aligned}$$

Proposition 7.1 was used in eq. (7.23). Sending $x\to+\infty$ and applying Proposition 7.2 yields the second equality of (4). □

Lemma 7.5. *Let* $\lambda,\mu>0$, *and* $\mathrm{Re}(s)>0$. *Then*

$$\int_0^\infty \mathcal{E}(\sqrt{\lambda t})\exp(-\mu t)t^s\frac{dt}{t} = \frac{1}{2}\pi^{-1/2}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\pi\lambda}{\mu};\frac{1}{2},\frac{1}{2}-s\right). \tag{7.25}$$

Proof. First of all, note that the left-hand side of Equation (7.25) converges: The integrand is $\exp(-O(t))$ as $t\to\infty$ and $O\left(t^{\mathrm{Re}\,s-\frac{1}{2}}\right)$ as $t\to 0$. Write $\mathcal{E}(\sqrt{\lambda t})=\frac{1}{2}\int_0^{\lambda t}u^{-1/2}e^{-\pi u}du$. The left-hand side of Equation (7.25) may be rewritten, using the substitution $u=\frac{\mu t v}{\pi}$ in the inner integral, as

$$\begin{aligned}\int_0^\infty \mathcal{E}(\sqrt{\lambda t})\exp(-\mu t)t^s\frac{dt}{t} &= \frac{1}{2}\int_0^\infty\int_0^{\lambda t}u^{-1/2}e^{-(\pi u+\mu t)}t^s\,du\frac{dt}{t} && (7.26)\\ &= \frac{1}{2}\int_0^\infty\int_0^{\frac{\pi\lambda}{\mu}}\left(\frac{\mu t v}{\pi}\right)^{-1/2}e^{-(\mu t v+\mu t)}t^s\,\frac{\mu t}{\pi}dv\frac{dt}{t}. && (7.27)\end{aligned}$$

The double integral is absolutely convergent (indeed, the integrand is nonnegative, and we already showed convergence), so we may swap the integrals. We compute

$$\begin{aligned}\int_0^\infty \mathcal{E}(\sqrt{\lambda t})\exp(-\mu t)t^s\frac{dt}{t} &= \frac{1}{2}\left(\frac{\mu}{\pi}\right)^{1/2}\int_0^{\frac{\pi\lambda}{\mu}}v^{-1/2}\left(\int_0^\infty e^{-\mu t(v+1)}t^{s+\frac{1}{2}}\,\frac{dt}{t}\right)dv && (7.28)\\ &= \frac{1}{2}\left(\frac{\mu}{\pi}\right)^{1/2}\int_0^{\frac{\pi\lambda}{\mu}}v^{-1/2}\left(\Gamma\left(s+\frac{1}{2}\right)(\mu(v+1))^{-(s+\frac{1}{2})}\right)dv && (7.29)\\ &= \frac{1}{2}\pi^{-1/2}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\int_0^{\frac{\pi\lambda}{\mu}}v^{-1/2}(v+1)^{-(s+\frac{1}{2})}\,dv && (7.30)\\ &= \frac{1}{2}\pi^{-1/2}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\pi\lambda}{\mu};\frac{1}{2},\frac{1}{2}-s\right). && (7.31)\end{aligned}$$

This proves Equation (7.25). □

Lemma 7.6. *Let* $\nu_1, \nu_2 \in \mathbb{R}$ *and* $\mu \in \mathbb{C}$ *satisfying* $\mathrm{Re}(\mu) > -\pi \max\{\nu_1^2, \nu_2^2\}$ *if* $\mathrm{sgn}(\nu_1) = \mathrm{sgn}(\nu_2)$ *and* $\mathrm{Re}(\mu) > 0$ *otherwise. Then,*

$$\begin{aligned}
&\int_0^\infty \mathcal{E}\left(\nu t^{1/2}\right)\Big|_{\nu=\nu_1}^{\nu_2} \exp(-\mu t) t^s \frac{dt}{t} \\
&= \frac{1}{2}\left(\mathrm{sgn}(\nu_2) - \mathrm{sgn}(\nu_1)\right)\Gamma(s)\mu^{-s} \\
&\quad - \frac{\mathrm{sgn}(\nu_2)}{2s}\pi^{-\left(s+\frac{1}{2}\right)}\Gamma\left(s+\frac{1}{2}\right)|\nu_2|^{-2s}\,{}_2F_1\left(s, s+\frac{1}{2}, s+1; -\frac{\mu}{\pi\nu_2^2}\right) \\
&\quad + \frac{\mathrm{sgn}(\nu_1)}{2s}\pi^{-\left(s+\frac{1}{2}\right)}\Gamma\left(s+\frac{1}{2}\right)|\nu_1|^{-2s}\,{}_2F_1\left(s, s+\frac{1}{2}, s+1; -\frac{\mu}{\pi\nu_1^2}\right).
\end{aligned} \tag{7.32}$$

Proof. Initially, consider $\lambda, \mu > 0$, as in Lemma 7.5. We have

$$\begin{aligned}
&\int_0^\infty \mathcal{E}\left(\sqrt{\lambda t}\right) \exp(-\mu t) t^s \frac{dt}{t} \\
&= \frac{1}{2}\pi^{-\frac{1}{2}}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\pi\lambda}{\mu}; \frac{1}{2}, \frac{1}{2}-s\right) && (7.33)\\
&= \frac{1}{2}\pi^{-\frac{1}{2}}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\left(\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{1}{2})} - \tilde{B}\left(\frac{\mu}{\pi\lambda}; s, 1-s\right)\right) && (7.34)\\
&= \frac{1}{2}\Gamma(s)\mu^{-s} - \frac{1}{2}\pi^{-\frac{1}{2}}\mu^{-s}\Gamma\left(s+\frac{1}{2}\right)\tilde{B}\left(\frac{\mu}{\pi\lambda}; s, 1-s\right) && (7.35)\\
&= \frac{1}{2}\Gamma(s)\mu^{-s} - \frac{1}{2s}\pi^{-\left(s+\frac{1}{2}\right)}\Gamma\left(s+\frac{1}{2}\right)\lambda^{-s}{}_2F_1\left(s, s+\frac{1}{2}, s+1; -\frac{\mu}{\pi\lambda}\right), && (7.36)
\end{aligned}$$

using parts (2) and (3) of Proposition 7.4. Equation (7.32) follows for positive real μ. But the integral on the left-hand side of eq. (7.32) converges for $\mathrm{Re}(\mu) > -\pi \max\{\nu_1^2, \nu_2^2\}$ if $\mathrm{sgn}(\nu_1) = \mathrm{sgn}(\nu_2)$ and $\mathrm{Re}(\mu) > 0$ otherwise, and both sides are analytic functions in μ on this domain. Thus, eq. (7.32) holds in general by analytic continuation. □

7.2. **The series expansion.** We are now ready to prove Theorem 1.2, which we first restate here for convenience.

Theorem 1.2. If $c_1, c_2 \in \mathbb{R}^g$, and $\mathrm{Re}(s) > 1$, then the indefinite zeta function may be written as

$$\hat{\zeta}_{p,q}^{c_1,c_2}(\Omega, s) = \pi^{-s}\Gamma(s)\zeta_{p,q}^{c_1,c_2}(\Omega, s) - \pi^{-\left(s+\frac{1}{2}\right)}\Gamma\left(s+\frac{1}{2}\right)\left(\xi_{p,q}^{c_2}(\Omega, s) - \xi_{p,q}^{c_1}(\Omega, s)\right), \tag{7.37}$$

where $M = \mathrm{Im}(\Omega)$,

$$\zeta_{p,q}^{c_1,c_2}(\Omega, s) = \frac{1}{2}\sum_{n\in\mathbb{Z}^g+q}\left(\mathrm{sgn}(c_1^\top M n) - \mathrm{sgn}(c_2^\top M n)\right) e\left(p^\top n\right) Q_{-i\Omega}(n)^{-s}, \tag{7.38}$$

and

$$\xi^{c}_{p,q}(\Omega,s) = \frac{1}{2}\sum_{\nu\in\mathbb{Z}^g+q} \operatorname{sgn}(c^\top Mn) e\left(p^\top n\right)\left(\frac{(c^\top Mn)^2}{Q_M(c)}\right)^{-s}$$
$$\cdot\, {}_2F_1\left(s, s+\frac{1}{2}, s+1; \frac{2Q_M(c)Q_{-i\Omega}(n)}{(c^\top Mn)^2}\right). \tag{7.39}$$

Proof. Take the Mellin transform of the theta series term-by-term, and apply Lemma 7.6. Note that the series for $\xi^c_{p,q}(\Omega,s)$ converges absolutely, so the series may be split up like this. □

The function $\zeta^{c_1,c_2}_{p,q}(\Omega,s)$ here is a Dirichlet series summed over a double cone, with any lattice points on the boundary of the cone weighted by $\frac{1}{2}$. The coefficients of the terms are $\pm e\left(p^\top n\right)$, where the sign is determined by whether one is in the positive or negative part of the double cone.

Theorem 7.7. *Suppose $(c_1, c_2, p+\Omega q, \Omega)$ is P-stable. Then, $\xi^{c_1}_{p,q}(\Omega,s) = \xi^{c_2}_{p,q}(\Omega,s)$ and $\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega,s) = \pi^{-s}\Gamma(s)\zeta^{c_1,c_2}_{p,q}(\Omega,s)$.*

Proof. The equality of the $\xi^{c_j}_{p,q}(\Omega,s)$ follows by the substitution $n\mapsto Pn$ and the definition of P-stability. The equation

$$\hat{\zeta}^{c_1,c_2}_{p,q}(\Omega,s) = \pi^{-s}\Gamma(s)\zeta^{c_1,c_2}_{p,q}(\Omega,s) \tag{7.40}$$

then follows from Theorem 1.2. □

8. Zeta functions of ray ideal classes in real quadratic fields

In this section, we will specialize indefinite zeta functions to obtain certain zeta functions to obtain certain zeta functions attached to real quadratic fields. We define two Dirichlet series, $\zeta(s,A)$ and $Z_A(s)$, attached to a ray ideal class A of the ring of integers of a number field.

Definition 8.1 (Ray class zeta function)**.** Let K be any number field and $\mathfrak{c}$ an ideal of the maximal order $\mathcal{O}_K$. Let S be a subset of the real places of K (i.e., the embeddings $K\hookrightarrow\mathbb{R}$). Let A be a ray ideal class modulo $\mathfrak{c}S$, that is, an element of the group

$$\operatorname{Cl}_{\mathfrak{c}S}(\mathcal{O}_K) := \frac{\{\text{nonzero fractional ideals of } \mathcal{O}_K \text{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K : a\equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}}. \tag{8.1}$$

Define the *zeta function of* A to be

$$\zeta(s,A) = \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}. \tag{8.2}$$

This function has a simple pole at $s=1$ with residue independent of A. The pole may be eliminated by considering the function $Z_A(s)$, defined as follows.

Definition 8.2 (Differenced ray class zeta function)**.** Let R be the element of $\operatorname{Cl}_{\mathfrak{c}S}$ defined by

$$R = \{a\mathcal{O}_K : a\equiv -1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at each place in } S\}. \tag{8.3}$$

Define the *differenced zeta function of* A to be

$$Z_A(s) = \zeta(s,A) - \zeta(s,RA). \tag{8.4}$$

The function $Z_A(s)$ is holomorphic at $s = 1$.

Now, specialize to the case where $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant D. Let $\mathcal{O}_K$ be the maximal order of K, and let $\mathfrak{c}$ be an ideal of $\mathcal{O}_K$. Let A be a narrow ray ideal class modulo $\mathfrak{c}$, that is, an element of the group $\mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}(\mathcal{O}_K)$. We show, as promised in the introduction, that the indefinite zeta function specializes to the L-series $Z_A(s)$ attached to a ray class of an order in a real quadratic field.

Theorem 1.3. For each $A \in \mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}$ and integral ideal $\mathfrak{b} \in A^{-1}$, there exists a real symmetric 2×2 matrix M, vectors $c_1, c_2 \in \mathbb{R}^2$,and $q \in \mathbb{Q}^2$ such that

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s). \tag{8.5}$$

Proof. The differenced zeta function $Z_A(s)$ is

$$Z_A(s) = \sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s} - \sum_{\mathfrak{a}\in RA} N(\mathfrak{a})^{-s}. \tag{8.6}$$

We have

$$N(\mathfrak{b})^{-s}Z_A(s) = \sum_{\mathfrak{a}\in A} N(\mathfrak{b}\mathfrak{a})^{-s} - \sum_{\mathfrak{a}\in RA} N(\mathfrak{b}\mathfrak{a})^{-s} \tag{8.7}$$

$$= \sum_{\substack{b\in\mathfrak{b} \\ (b)\in I \\ \text{up to units}}} N(b)^{-s} - \sum_{\substack{b\in\mathfrak{b} \\ (b)\in R \\ \text{up to units}}} N(b)^{-s}. \tag{8.8}$$

Write $\mathfrak{b}\mathfrak{c} = \gamma_1\mathbb{Z}+\gamma_2\mathbb{Z}$. The norm form $N(n_1\gamma_1+n_2\gamma_2) = Q_M\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ for some real symmetric matrix M with integer coefficients. The signature of M is $(1, 1)$, just like the norm form for K. Since $\mathfrak{b}$ and $\mathfrak{c}$ are relatively prime (meaning $\mathfrak{b} + \mathfrak{c} = \mathcal{O}_K$), there exists by the Chinese remainder theorem some $b_0 \in \mathcal{O}_K$ such that $b \equiv b_0 \pmod{\mathfrak{b}\mathfrak{c}}$ if and only if $b \equiv 0 \pmod{\mathfrak{b}}$ and $b \equiv 1 \pmod{\mathfrak{c}}$. Express $b_0 = p_1\gamma_1+p_2\gamma_2$ for rational numbers p_1, p_2, and set $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$.

Let ε_0 be the fundamental unit of $\mathcal{O}_K$, and let ε ($= \varepsilon_0^k$ for some k) be the smallest totally positive unit of $\mathcal{O}_K$ greater than 1 such that $\varepsilon \equiv 1 \pmod{\mathfrak{c}}$.

Choose any $c_1 \in \mathbb{R}^2$ such that $Q_M(c_1) < 0$. Let P be the matrix describing the linear action of ε on $\mathfrak{b}$ by multiplication, i.e., $\varepsilon(\beta^\top n) = \beta^\top(Pn)$. Set $c_2 = Pc_1$.

Thus, we have

$$N(\beta)^{-s}Z_A(s) = \frac{1}{2}\sum_{n\in\mathbb{Z}^2+q} \left(\operatorname{sgn}(c_2^\top Mn) - \operatorname{sgn}(c_1^\top Mn)\right) Q_M(n). \tag{8.9}$$

Moreover, (c_1, c_2, p, Ω) is P-stable. So, by Theorem 7.7, eq. (8.9) may be rewritten as

$$(2\pi N(\mathfrak{b}))^{-s}\Gamma(s)Z_A(s) = \hat{\zeta}_{0,q}^{c_1,c_2}(iM, s), \tag{8.10}$$

completing the proof. □

8.1. **Example.** Let $K = \mathbb{Q}(\sqrt{3})$, so $\mathbf{O}_K = \mathbb{Z}[\sqrt{3}]$, and let $\mathfrak{c} = 5\mathbf{O}_K$. The ray class group $\mathrm{Cl}_{\mathfrak{c}\infty_2} \cong \mathbb{Z}/8\mathbb{Z}$. The fundamental unit $\varepsilon = 2 + \sqrt{3}$ is totally positive: $\varepsilon\varepsilon' = 1$. It has order 3 modulo 5: $\varepsilon^3 = 26 + 15\sqrt{3} \equiv 1 \pmod 5$. In this section, we use the analytic continuation eq. (6.11) for indefinite zeta functions to compute $Z_I'(0)$, where I is the principal ray class of $\mathrm{Cl}_{\mathfrak{c}\infty_2}$.

By definition, $Z_I = \zeta(s, I) - \zeta(s, R)$ where

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } a \text{ is positive at } \infty_2\} \tag{8.11}$$
$$= \{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ is negative at } \infty_2\}. \tag{8.12}$$

Write $I = I_+ \sqcup I_-$ and $R = R_+ \sqcup R_-$, where $I_\pm$ and $R_\pm$ are the following ray ideal classes in $\mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}$:

$$I_\pm = \{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ has sign } \pm \text{ at } \infty_1 \text{ and } + \text{ at } \infty_2\}, \tag{8.13}$$
$$R_\pm = \{a\mathcal{O}_K : a \equiv 1 \pmod{\mathfrak{c}} \text{ and } a \text{ has sign } \pm \text{ at } \infty_1 \text{ and } - \text{ at } \infty_2\}. \tag{8.14}$$

Thus, $Z_I(s) = \zeta(s, I_+) + \zeta(s, I_-) - \zeta(s, R_+) - \zeta(s, R_-)$. The Galois automorphism $(a_1 + a_2\sqrt{3})^\sigma = (a_1 - a_2\sqrt{3})$ defines a norm-preserving bijection between I_- and R_+, so the middle terms cancel and

$$Z_I(s) = \zeta(s, I_+) - \zeta(s, R_-) = Z_{I_+}(s). \tag{8.15}$$

To the principal ray class I_+ of $\mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2}$, we associate $\Omega = iM$ where $M = \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}$ and $q = \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}$. We may choose $c_1 \in \mathbb{R}^2$ arbitrarily so long as $c_1^\top M c_1 < 0$; take $c_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The left action of ε on $\mathbb{Z} + \sqrt{3}\mathbb{Z}$ is given by the matrix $P = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. By Theorem 1.3,

$$(2\pi)^{-s}\Gamma(s) Z_{I_+}(s) = \hat{\zeta}_{0,q}^{c_1, P^3 c_1}(iM, s). \tag{8.16}$$

Taking a limit as $s \to 0$, and using eq. (8.15), eq. (8.16) becomes

$$Z_I'(0) = Z_{I_+}'(s) = \hat{\zeta}_{0,q}^{c_1, P^3 c_1}(iM, 0). \tag{8.17}$$

For the purpose of making the numerical computation more efficient, we split up the right-hand side as

$$Z_I'(0) = \hat{\zeta}_{0,q}^{c_1, P c_1}(iM, 0) + \hat{\zeta}_{0,q}^{P c_1, P^2 c_1}(iM, 0) + \hat{\zeta}_{0,q}^{P^2 c_1, P^3 c_1}(iM, 0) \tag{8.18}$$
$$= \hat{\zeta}_{0,q_0}^{c_1, P c_1}(iM, 0) + \hat{\zeta}_{0,q_1}^{c_1, P c_1}(iM, 0) + \hat{\zeta}_{0,q_2}^{c_1, P c_1}(iM, 0), \tag{8.19}$$

where $q_0 = q = \frac{1}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q_1 = q = \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $q_2 = q = \frac{1}{5}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ are obtained from the residues of $\varepsilon^0, \varepsilon^1, \varepsilon^2$ modulo 5.

Using eq. (6.11), we computed $Z_I'(0)$ to 100 decimal digits. The decimal begins

$$Z_I'(0) = 1.3586306533922081625951130823\ldots. \tag{8.20}$$

The conjectural Stark unit is $\exp(Z_I'(0)) = 3.8908617139430792553376439596\ldots$. We used the `RootApproximant[]` function in Mathematica, which uses lattice basis reduction internally, to find a degree 16 integer polynomial having this number as a root, and we factored that polynomial over $\mathbb{Q}(\sqrt{3})$. To 100 digits, $\exp(Z_I'(0))$ is equivalent to be the root of the polynomial

$$x^8 - (8 + 5\sqrt{3})x^7 + (53 + 30\sqrt{3})x^6 - (156 + 90\sqrt{3})x^5 + (225 + 130\sqrt{3})x^4$$
$$- (156 + 90\sqrt{3})x^3 + (53 + 30\sqrt{3})x^2 - (8 + 5\sqrt{3})x + 1. \tag{8.21}$$

We have verified that this root generates the expected class field H_2.

We have also computed $Z_I'(0)$ a different way in PARI/GP, using its internal algorithms for computing Hecke L-values. We obtained the same numerical answer this way.

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