Complex equiangular lines and the Stark conjectures

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Unit distance graphs in metric spaces

Definition

A unit distance embedding of an undirected graph $G = (V, E)$ *into a metric space* X *is an injection* $i : V \to X$ *such that for some constant* $c > 0$ *and for every edge* $\{v, w\} \in E$*, the distance* $\mathrm{dist}(i(v), i(w)) = c$*.*

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Question

Given a metric space X*, what is the largest* $n = \kappa(X)$ *such that the complete graph* K_n *on* n *vertices has a unit distance embedding into* X*?*

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Proof.

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Finally, there is no point of $\mathbb{R}^d$ of unit distance to the vertices of Δ_d, so K_{d+2} cannot embed into $\mathbb{R}^d$. □

The metric on projective space

A metric on real or complex projective space may be defined in terms of the angles between the lines.

Definition

For a pair of lines $\mathbb{C}v, \mathbb{C}w \in \mathbb{P}^{d-1}(\mathbb{C})$ *represented by unit vectors* v, w*, the* ***angle*** *between* $\mathbb{C}v$ *and* $\mathbb{C}w$ *is*

$$\angle(v, w) = \arccos\left(|\langle v, w\rangle|\right),$$

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A unit distance embedding of a complete graph into a projective space is also called a set of equiangular lines.

Unit distance complete graphs in projective spaces

Theorem (Delsarte, Goethals, and Seidel, 1975)

$\kappa(\mathbb{P}^{d-1}(\mathbb{R})) \leq \frac{d(d+1)}{2}$ *and* $\kappa(\mathbb{P}^{d-1}(\mathbb{C})) \leq d^2$.

Proof.

Represent elements of projective space by unit column vectors. The map

$$v \mapsto vv^\dagger$$

sends v to the $d \times d$ matrix defining "Hermitian projection onto $\mathbb{C}v$".

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SICs

Definition

A SIC (SIC-POVM; symmetric informationally complete positive operator-valued measure) is (a generalised quantum measurement equivalent to) a set of d^2 equiangular lines in $\mathbb{C}^d$. Formally, for a set of equiangular lines $\{\mathbb{C}v_1, \ldots, \mathbb{C}v_{d^2}\} \subset \mathbb{P}^{d-1}(\mathbb{C})$, the associated SIC-POVM is the set of rank 1 Hermitian matrices $\left\{\frac{1}{d}v_1v_1^\dagger, \ldots, \frac{1}{d}v_{d^2}v_{d^2}^\dagger\right\}$.

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For unit vectors $\{v_1, \ldots, v_{d^2}\}$ defining a SIC, $\left|\langle v_i, v_j\rangle\right| = \frac{1}{\sqrt{d+1}}$ for $i \neq j$.

Conjecture (Zauner, 1999)

SICs exist in every dimension d. That is, $\kappa(\mathbb{P}^{d-1}(\mathbb{C})) = d^2$.

Applications and mathematical significance of SICs

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- and quantum foundations, specifically the theory of quantum Bayesianism or QBism.
- SICs also arise as maximal equiangular tight frames...
- and as minimal complex spherical 2-designs.
- SICs (and generalisations thereof) are sometimes called “line packings” to draw an analogy with sphere packings.

Results

Form of main result

Stark conjectures
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Form of main result

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This result is a reduction of:

- A problem in frame theory to a problem in number theory.
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Caveats:

- Valid for prime dimension $d \equiv 2 \pmod 3$ (general d is work in progress).
- “Further hypotheses” are somewhat artificial.

Results

Alternative form of main result

Conjectural construction of SICs

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New practical algorithm for constructing SICs using L-functions
(valid in prime dimensions $d \equiv 2 \pmod{3}$)

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Alternative form of main result

Conjectural construction of SICs

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New practical algorithm for constructing SICs using L-functions (valid in prime dimensions $d \equiv 2 \pmod{3}$)

The algorithm is used to give the first construction of an exact SIC in dimension $d = 23$.

Ray class groups and ray class fields

Let K be a number field and $\mathcal{O}_K$ its ring of integers (maximal order). Let $\mathfrak{c}$ be a ideal in $\mathcal{O}_K$, and let S be a subset of the real embeddings of K.

Definition (Ray class group modulo $\mathfrak{c}, S$)

$$\mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K) = \frac{\{\textit{fractional ideals of } \mathcal{O}_K \textit{ coprime to } \mathfrak{c}\}}{\{a\mathcal{O}_K \textit{ s.t. } a \equiv 1 \pmod{\mathfrak{c}} \textit{ and } \rho(a) > 0 \textit{ for } \rho \in S\}}$$

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Class field theory associates to $\mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$ a ray class field $L_{\mathfrak{c},S}$, an abelian extension of K with Galois group $\mathrm{Gal}(L_{\mathfrak{c},S}/K) = \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$. Varying $\mathfrak{c}$ and S, the ray class fields are cofinal among all abelian extensions of K.

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- Given any base field ("realm of rationality"), Hilbert wanted "analytic functions" that play the role of $e(z)$.
- Harold Stark conjectured in a series of papers (1971–1980) that $\exp(cZ'(1))$, for certain linear combinations $Z(s)$ of Hecke L-functions of K, generate abelian extensions of K.

L-functions at $s = 1$: rational example

The following formula can be proved using calculus. Try it!

Example

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + \frac{1}{15} + - - + \cdots = \frac{1}{\sqrt{2}} \log\left(1 + \sqrt{2}\right)$$

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The left-hand side is the value $L(1, \chi)$, where $\chi(n) = \left(\frac{2}{n}\right)$ is the Dirichlet character associated to the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. The right-hand side involves $\varepsilon = 1 + \sqrt{2}$, the fundamental unit of $\mathbb{Q}(\sqrt{2})$.

L-functions at $s = 1$: imaginary quadratic example

The following formula is proved using the theory of complex multiplication for elliptic curves. The notation $e(z) := e^{2\pi i z}$.

Example

$$\sum\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{e(m/5) - e(2m/5)}{m^2 + mn + n^2} = \frac{2\pi}{\sqrt{3}} \log\left(\varepsilon^{1/5}\right)$$

where $\varepsilon = 29 + 12\sqrt{5} + 2\sqrt{6(65 + 29\sqrt{5})}$.

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This example is related to the 5-torsion points of the elliptic curve $y^2 = x^3 + 1$. This elliptic curve has "complex multiplication by $\mathbb{Z}[\omega]$" ($\omega = \frac{-1+\sqrt{-3}}{2}$), because of the extra endomorphism $(x, y) \mapsto (\omega x, y)$.

L-functions at $s = 1$: real quadratic example

The following formula is an open conjecture!

Example

$$\sum_{m=1}^{\infty} \sum_{\substack{n\in\mathbb{Z} \\ -\frac{5}{3}m\le n<\frac{5}{3}m}} \frac{e(4m/5) - e(m/5)}{3m^2 - n^2} = \frac{\pi}{i\sqrt{3}} \log(\varepsilon),$$

where $\varepsilon \approx 3.890861714$ *is a root of the polynomial equation*

$$x^8 - (8+5\sqrt{3})x^7 + (53+30\sqrt{3})x^6 - (156+90\sqrt{3})x^5 + (225+130\sqrt{3})x^4 - (156+90\sqrt{3})x^3 + (53+30\sqrt{3})x^2 - (8+5\sqrt{3})x + 1 = 0.$$

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The number ε is an algebraic unit in the ray class field of $\mathbb{Q}(\sqrt{3})$ modulo $5\infty_2$. This conjecture is part of the Stark conjectures.

Zeta functions associated to ray classes

Definition

For $A \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$, the associated zeta function is

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Let $R \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$ be the ideal class

$$R = \{a\mathcal{O}_K : a \equiv -1 \pmod{\mathfrak{c}} \text{ and } \rho(a) > 0 \text{ for } \rho \in S\}.$$

Definition

For $A \in \mathrm{Cl}_{\mathfrak{c},S}(\mathcal{O}_K)$, the associated differenced zeta function is

$$Z_A(s) = \zeta(s, A) - \zeta(s, RA).$$

Rank 1 abelian Stark conjecture over a real quadratic field

Conjecture (Stark, 1976)

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Then,

(1) $Z_A'(0) = \log(\rho_1(\varepsilon_A))$ *for a unit* $\varepsilon_A \in H_2$.

(2) *The units ε_A are compatible with the Artin map* $\mathrm{Art} : \mathrm{Cl}_{\mathfrak{c}\infty_1\infty_2} \to \mathrm{Gal}(H_2/K)$. *Specifically,* $\varepsilon_A = \varepsilon_I{}^{\mathrm{Art}(A)}$.

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- The ray class group $\mathrm{Cl}_{\mathfrak{c}\infty_2} \cong \mathbb{Z}/8\mathbb{Z}$. Let I be the identity.
- We can calculate $Z_I'(0) \approx 1.3586306534$ and $\exp(Z_I'(0)) \approx 3.8908617139$—apparently the root of a degree 8 polynomial.

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- The equations defining the SIC condition are $\approx d^4$ quartic equations in $\approx d^2$ variables. Groebner basis computation becomes too difficult for $d \geq 5$.

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- Studying group covariant SICs (for G fixed) reduces the conditions to $\approx d^2$ quartic equations in $\approx d$ variables.

Heisenberg SICs

All but one of the known SICs are (Weyl-)Heisenberg (covariant) SICs: The orbit of a single fiducial vector v under the discrete (Weyl-)Heisenberg group $\mathrm{H}(d) = \langle \zeta_d^{\frac{d+1}{2}} I, X, Z \rangle$;

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}; \quad Z = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_d & 0 & \cdots & 0 \\ 0 & 0 & \zeta_d^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_d^{d-1} \end{pmatrix}.$$

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- Displacement operators $D_{m,n} = \zeta_d^{\frac{d+1}{2}mn} X^m Z^n$, $0 \leq m, n < d$, are preferred coset reps of $H(d)/\langle \zeta_d^{\frac{d+1}{2}} I \rangle$.
- Overlap phases of a Heisenberg SIC are $\sqrt{d+1}\langle v, D_{m,n} v \rangle$ with $(m,n) \neq (0,0)$.

Known results on SIC existence

Refined version of Zauner's conjecture:

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- Exact algebraic solutions in dimensions 1–21, 23, 24, 28, 30, 31, 35, 37, 39, 43, 48, 53, 124, 195, and 323. (Marcus Grassl reports solutions in 31 additional dimensions!) Numerical (probable) solutions in every dimension up to 151 and several other dimensions up to 844.

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- Surprising observation: in known examples, the field of definition of Heisenberg SICs in dimension $d \geq 4$ is an abelian extension of $K = \mathbb{Q}(\sqrt{(d+1)(d-3)})$, often a particular ray class field $L_{(d)\infty_1}$ (Appleby, Flammia, McConnell, and Yard; 2016).

An observation when $d = 5$

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For an appropriate choice of fiducial vector, the squares of the overlap phases (each having multiplicity 3) of a Heisenberg SIC in dimension $d = 5$ are the roots of the polynomial

$$x^8 - (8 - 5\sqrt{3})x^7 + (53 - 30\sqrt{3})x^6 - (156 - 90\sqrt{3})x^5 + (225 - 130\sqrt{3})x^4 - (156 - 90\sqrt{3})x^3 + (53 - 30\sqrt{3})x^2 - (8 - 5\sqrt{3})x + 1 = 0.$$

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Might this observation generalise? Yes! Squares of overlap phases are Galois conjugate to powers of Stark units in all the cases I've checked, and this has been made totally explicit in the case of d an odd prime 2 modulo 3.

Conjectures

Conjecture 1 (K; existence of special units in ray class field)

Let $d \equiv 2 \pmod 3$ *be an odd prime. Let* $\Delta = (d+1)(d-3)$ *and* $K = \mathbb{Q}(\sqrt{\Delta})$*. With indices* $m, n \in \mathbb{Z}/d\mathbb{Z}$*, let*

$$A_{m,n} = \{\alpha\mathcal{O}_K : \alpha \equiv m + n\sqrt{\Delta} \pmod d \text{ and } \rho_2(\alpha) > 0\} \in \mathrm{Cl}_{(d)\infty_2}.$$

Then, there is a real algebraic unit α *such that the ray class field* $L_{(d)\infty_2} = K(\alpha)$ *and* $\alpha_{m,n} := \alpha^{\mathrm{Art}(A_{m,n})}$ *satisfy:*

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(4) *Fix* $\mathfrak{p} | d\mathcal{O}_{L_{(d)\infty_2}}$, *and let* $(d+1){\nu_{m,n}}^2 = \alpha_{m,n}$ *satisfying* $\nu_{m,n} \equiv 1 \pmod{\mathfrak{p}}$. *Let* $\nu_{0,0} = 1$. *Then, the matrix*

$$M = \frac{1}{d}\sum_{m=0}^{d-1}\sum_{n=0}^{d-1} \nu_{m,n} D_{-m,-n} \quad \text{...is a rank 1 idempotent.}$$

Conjectures

Conjecture 2 (K; find special units as Stark units)

A unit α satisfying Conjecture 1 and its Galois conjugates over K may be constructed as Stark units

$$\alpha^{\mathrm{Art}(A)} = \exp\left(Z_A'(0)\right),$$

for all $A \in \mathrm{Cl}_{(d)\infty_2}$.

As before, the differenced ray class zeta function $Z_A(s)$ is defined as

$$Z_A(s) = \left(\sum_{\mathfrak{a}\in A} N(\mathfrak{a})^{-s}\right) - \left(\sum_{\mathfrak{a}\in RA} N(\mathfrak{a})^{-s}\right),$$

where $R = \{a\mathcal{O}_K : a \equiv -1 \pmod{d} \text{ and } \rho_2(a) > 0\}$.

Results

Theorem (K)

Let d be an odd prime such that $d \equiv 2 \pmod{3}$. Assume Conjecture 1, and let M be the matrix constructed therein. Let $\sigma \in \mathrm{Gal}(L_{(d)\infty_2}/\mathbb{Q})$ be any Galois automorphism not fixing K; that is, $\sigma(\sqrt{\Delta}) = -\sqrt{\Delta}$. Then $\sigma(M) = vv^\dagger$ for a fiducial vector v of a Heisenberg SIC.

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- The Stark unit construction of Conjecture 2 works (numerically) at least for $d = 5, 11, 17$, and 23 ($d = 53$ has been “spot-checked”).

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- The Stark unit construction of Conjecture 2 works (numerically) at least for $d = 5, 11, 17$, and 23 ($d = 53$ has been "spot-checked").
- After finding the corresponding exact units by lattice basis reduction, we provide the first exact construction of a SIC in dimension 23.

Thank you!

Thank you to the organisers!

Kopp, Gene. SICs and the Stark conjectures. Preprint available at arxiv:1807.05877. To appear in *Int. Math. Res. Notices*.