The Arithmetic Geometry of Resonant Rossby Wave Triads

Gene Kopp

University of Michigan

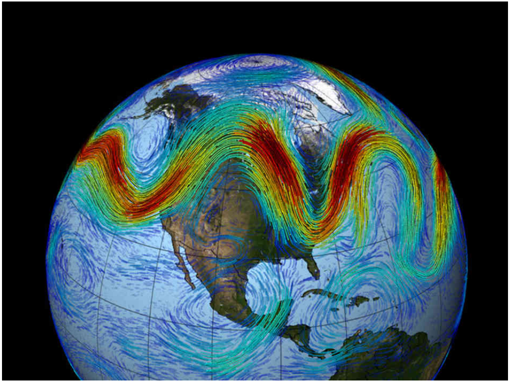
http://www-personal.umich.edu/~gkopp/

AMS Fall Central Sectional Meeting
University of St. Thomas
October 29, 2016

Part One: Rossby Waves and Resonance

Rossby Waves

- Atmospheric Rossby waves on Earth are large-scale meanders in high-altitude winds. They are a major influence on the weather.



Rossby Waves

- Mathematically, Rossby waves are solutions to the Charney-Hasegawa-Mima equation (CHME)...

$$\frac{\partial}{\partial t}\left(\Delta\psi - F\psi\right) + \beta\frac{\partial\psi}{\partial x} + [\psi, \Delta\psi] = 0,$$

Rossby Waves

- Mathematically, Rossby waves are solutions to the Charney-Hasegawa-Mima equation (CHME)...

$$\frac{\partial}{\partial t}(\Delta\psi - F\psi) + \beta\frac{\partial\psi}{\partial x} + [\psi, \Delta\psi] = 0,$$

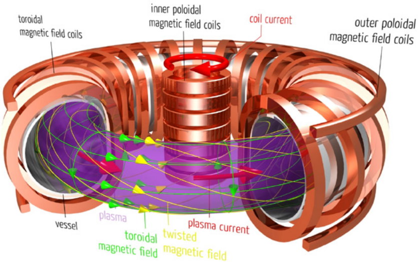
- with periodic boundary conditions.

Rossby Waves

- Mathematically, Rossby waves are solutions to the Charney-Hasegawa-Mima equation (CHME)...

$$\frac{\partial}{\partial t}(\Delta\psi - F\psi) + \beta\frac{\partial\psi}{\partial x} + [\psi, \Delta\psi] = 0,$$

- with periodic boundary conditions.
- The same PDE and boundary conditions describe drift waves in plasma in a tokamak.



Number theory and differential equations

- Why does number theory appear in physics and differential equations?

Number theory and differential equations

- Why does number theory appear in physics and differential equations?
- Periodic boundary conditions

Number theory and differential equations

- Why does number theory appear in physics and differential equations?
- Periodic boundary conditions
- Example from quantum mechanics: Bohr’s hydrogen atom. Integral number of modes for the electron.

Number theory and differential equations

- Why does number theory appear in physics and differential equations?
- Periodic boundary conditions
- Example from quantum mechanics: Bohr's hydrogen atom. Integral number of modes for the electron.
- On a circle, wavenumber $= 2\pi/$(frequency) is an integer. On a torus, wavenumbers are a vector in $\mathbb{Z}^2$.

Number theory and differential equations

- Why does number theory appear in physics and differential equations?
- Periodic boundary conditions
- Example from quantum mechanics: Bohr's hydrogen atom. Integral number of modes for the electron.
- On a circle, wavenumber $= 2\pi/(\text{frequency})$ is an integer. On a torus, wavenumbers are a vector in $\mathbb{Z}^2$.
- In the β-plane model of Rossby waves, there is a zonal (east/west) wavenumber a and a meridional (north/south) wavenumber b.

Resonances of Rossby Waves

- Under certain physical conditions (large β), exact resonances of triples of Rossby waves dominate the behavior of the system. This is a theorem of Yamada and Yoneda [YY].

Resonances of Rossby Waves

- Under certain physical conditions (large β), exact resonances of triples of Rossby waves dominate the behavior of the system. This is a theorem of Yamada and Yoneda [YY].
- The problem of finding resonant triads has been widely studied, and there are standard analytical methods.

Resonances of Rossby Waves

- Under certain physical conditions (large β), exact resonances of triples of Rossby waves dominate the behavior of the system. This is a theorem of Yamada and Yoneda [YY].
- The problem of finding resonant triads has been widely studied, and there are standard analytical methods.
- Bustamante and Hayat [BH] were the first to classify these triads algebraically, and they give a much quicker algorithm for enumerating them.

Resonances of Rossby Waves

- Under certain physical conditions (large β), exact resonances of triples of Rossby waves dominate the behavior of the system. This is a theorem of Yamada and Yoneda [YY].
- The problem of finding resonant triads has been widely studied, and there are standard analytical methods.
- Bustamante and Hayat [BH] were the first to classify these triads algebraically, and they give a much quicker algorithm for enumerating them.
- We [K] give a new algebraic classification and an even quicker algorithm.

Part Two: Application of Number Theory

Resonance Equation

For a wavenumber vector $(a, b) \in \mathbb{Z}^2$, the angular frequency is given by the dispersion relation

$$\omega(a, b) = -\frac{\beta a}{a^2 + b^2}.$$

Resonance Equation

For a wavenumber vector $(a, b) \in \mathbb{Z}^2$, the angular frequency is given by the dispersion relation

$$\omega(a, b) = -\frac{\beta a}{a^2 + b^2}.$$

A resonant triad consists of wavenumbers (a, b), $(x - a, y - b)$, and (x, y), satisfying the equation

$$\omega(a, b) + \omega(x - a, y - b) = \omega(x, y)$$
$$\frac{a}{a^2 + b^2} + \frac{x - a}{(x - a)^2 + (y - b)^2} = \frac{x}{x^2 + y^2}$$

Resonance Equation

For a wavenumber vector $(a, b) \in \mathbb{Z}^2$, the angular frequency is given by the dispersion relation

$$\omega(a, b) = -\frac{\beta a}{a^2 + b^2}.$$

A resonant triad consists of wavenumbers (a, b), $(x - a, y - b)$, and (x, y), satisfying the equation

$$\omega(a, b) + \omega(x - a, y - b) = \omega(x, y)$$
$$\frac{a}{a^2 + b^2} + \frac{x - a}{(x - a)^2 + (y - b)^2} = \frac{x}{x^2 + y^2}$$

Resonance Equation

$$x(a^2 + b^2)(a^2 + b^2 - 2ax - 2by) = a(x^2 + y^2)(x^2 + y^2 - 2ax - 2by)$$

Resonance Equation

- The resonance equation defines a degree five surface $X \subset \mathbb{P}^3$.
- Call $C(a, b)$ the curve defined by the resonance equation for fixed $(a, b) \in \mathbb{Z}$. This curve is defined over $\mathbb{Z}$, and $C(a, b)_{\mathbb{Q}}$ is the fiber of the map

$$\begin{aligned} X &\to \mathbb{P}^1 \\ [a : b : x : y] &\mapsto [a : b]. \end{aligned}$$

Resonance Equation

$$x(a^2 + b^2)(a^2 + b^2 - 2ax - 2by) = a(x^2 + y^2)(x^2 + y^2 - 2ax - 2by)$$

Results

Theorem

X is birational to $\mathbb{P}^2$ over $\mathbb{Q}$.

A generically one-to-one rational parametrization is given.

Results

Theorem

X is birational to $\mathbb{P}^2$ over $\mathbb{Q}$.

A generically one-to-one rational parametrization is given.

Theorem

For $(a, b) \in \mathbb{Z}$, $C(a, b)$ is birational to an elliptic curve with torsion $\mathbb{Z}/2\mathbb{Z}$ and rank ≥ 1, except when $a = 0$ or $b = 0$.

As a corollary, for any (a, b) with $ab \neq 0$, we can find $C(an, bn)$ with arbitrarily many integer points.

Birational Equivalence of X with $\mathbb{P}^2$

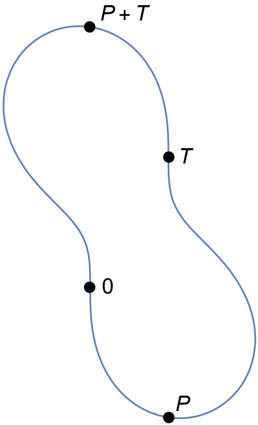
A rational parametrization $\mathbb{P}^2 \to X$, $[s : t : u] \mapsto [a : b : x : y]$

$$\begin{bmatrix} a \\ \vdots \\ b \\ \vdots \\ x \\ \vdots \\ y \end{bmatrix} = \begin{bmatrix} s^3 t(s-2u) \\ \vdots \\ s(-s^2u(s-2u)+(t^2+u^2)(t^2-2su+u^2)) \\ \vdots \\ t(t^2+u^2)(t^2-2su+u^2) \\ \vdots \\ (t^2+u^2)(-s^2(s-2u)+u(t^2-2su+u^2)) \end{bmatrix}$$

Its rational inverse $X \to \mathbb{P}^2$, $[a : b : x : y] \mapsto [s : t : u]$.

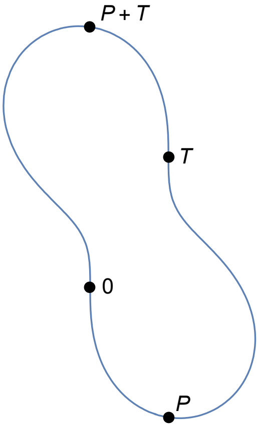
$$\begin{bmatrix} s \\ \vdots \\ t \\ \vdots \\ u \end{bmatrix} = \begin{bmatrix} a^2+b^2 \\ \vdots \\ bx-ay \\ \vdots \\ ax+by \end{bmatrix}$$

A Fibration by Elliptic Curves



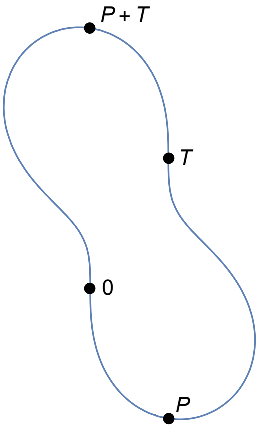
- The real points of $C(a, b)$ form a smooth closed loop, so there are finitely many integer points.

A Fibration by Elliptic Curves



- The real points of $C(a, b)$ form a smooth closed loop, so there are finitely many integer points.
- $C(a, b)$ is genus one, unless $[a : b] = [0 : 1], [\pm i : 1], [\pm 2i : 1]$, in which cases it is genus zero.

A Fibration by Elliptic Curves

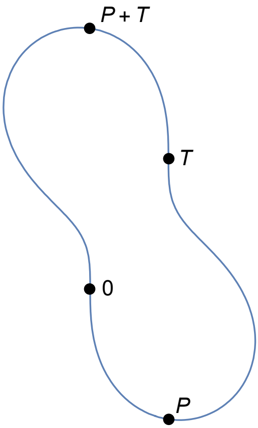


- The real points of $C(a,b)$ form a smooth closed loop, so there are finitely many integer points.
- $C(a,b)$ is genus one, unless $[a:b] = [0:1], [\pm i:1], [\pm 2i:1]$, in which cases it is genus zero.
- $C(a,b)$ has two (non-real) singularities. We normalize the curve to obtain a smooth model, and convert to Weierstrass form.

Weierstrass form of smooth model of $C(a,b)$

$$W^2 = Z^3 + (a^2 - 2b^2)Z^2 + (a^2 + b^2)^2 Z.$$

A Fibration by Elliptic Curves

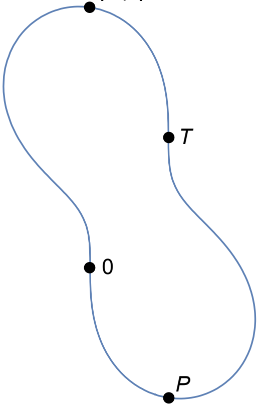


- There are four “trivial” integer points on $C(a, b)$, giving rise to zonal resonances and unstable single-wave “resonances.”

Weierstrass form of smooth model of $C(a, b)$

$$W^2 = Z^3 + (a^2 - 2b^2)Z^2 + (a^2 + b^2)^2 Z.$$

A Fibration by Elliptic Curves



- There are four "trivial" integer points on $C(a, b)$, giving rise to zonal resonances and unstable single-wave "resonances."
- The point $(0, 0)$ is taken to the identity of this elliptic curve. $T = (a, b)$ has order 2, $P = (0, 2b)$ has infinite order, and $(a, -b) = P + T$.

Weierstrass form of smooth model of $C(a, b)$

$$W^2 = Z^3 + (a^2 - 2b^2)Z^2 + (a^2 + b^2)^2 Z.$$

Computational Power

- How might all this algebraic geometry help us with mathematical modeling?

Computational Power

- How might all this algebraic geometry help us with mathematical modeling?
- Plug triples of integers into our parametrization and clear common factors to enumerate all triads.
- While we don't yet have good bounds on how long it will take to enumerate all triads up to a given wavenumber bound, in practice we have enumerated all 463×24 triads—more than found in the literature—up to wavenumber 5000 in 80 minutes on a MacBook Pro.

Computational Power

- How might all this algebraic geometry help us with mathematical modeling?
- Plug triples of integers into our parametrization and clear common factors to enumerate all triads.
- While we don't yet have good bounds on how long it will take to enumerate all triads up to a given wavenumber bound, in practice we have enumerated all 463×24 triads—more than found in the literature—up to wavenumber 5000 in 80 minutes on a MacBook Pro.
- We've also enumerated the first few resonant wavevectors with zonal group velocity zero: $(n, \pm n)$ for integers n in the sequence 13, 229, 3277, 504613, 155870857, 34589637433, 58803854910601, ... (This computation took a trivial amount of time.)

Thank you!

Thank you to the organizers!

Kopp, G. The arithmetic geometry of resonant Rossby wave triads. Submitted, 2016. arxiv:1605.04637.

Definition of the Wavenumber Set

Wavenumber Set

$$\Lambda := \{(a,b) \in \mathbb{Z}^2 : C(a,b) \text{ has a non-trivial integer point}\}.$$

$$\Lambda_{\text{new}} := \{(a,b) \in \mathbb{Z}^2 : C(a,b) \text{ has a new non-trivial integer point}\}.$$

Definition of the Wavenumber Set

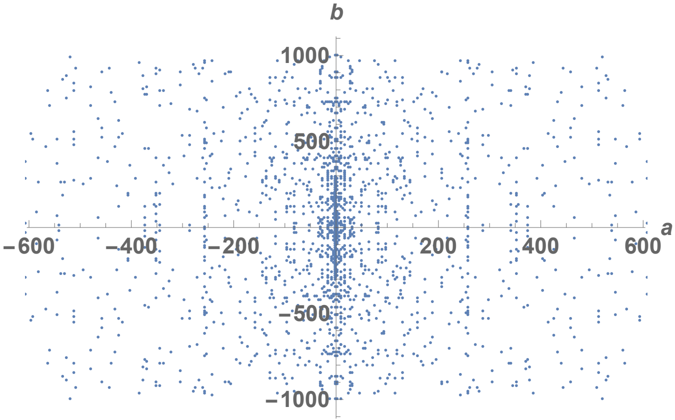
Wavenumber Set

$$\Lambda := \{(a,b) \in \mathbb{Z}^2 : C(a,b) \text{ has a non-trivial integer point}\}.$$

$$\Lambda_{\text{new}} := \{(a,b) \in \mathbb{Z}^2 : C(a,b) \text{ has a new non-trivial integer point}\}.$$

- non-trivial $(x,y) \in C(a,b)$: $x \neq 0$ and $x - a \neq 0$.
- new: $\gcd(a,b,x,y) = 1$.

Plot of the Wavenumber Set Λ_{new}



M. Bustamante and U. Hayat. Complete classification of discrete resonant Rossby/drift wave triads on periodic domains. *Commun. Nonlinear Sci. Numer. Simulat*, 18: 2402–2419, 2013.

D. Coumou, J. Lehmann, and J. Beckmann. The weakening summer circulation in the Northern Hemisphere mid-latitudes. *Science*, 348(6232): 324-327, 2015.

A. Kartashov and E. Kartashova. Discrete exact and quasi-resonances of Rossby/drift waves on β-plane with periodic boundary conditions. Preprint, arXiv:1307.8272v1, 2013.

N. Kishimoto and T. Yoneda. A number theoretical observation of a resonant interaction of Rossby waves. Preprint, arXiv:1409.1031v1, 2014.

G. Kopp. The arithmetic geometry of resonant Rossby wave triads. Submitted, 2016. arxiv:1605.04637.

V. Petoukhov, S. Rahmstorf, S. Petri, and H. Schellnhuber. Quasiresonant amplification of planetary waves and recent Northern Hemisphere weather extremes. *Proceedings of the National Academy of Sciences*, 110(14): 5336–5341, 2013.

M. Yamada and T. Yoneda. Resonant interaction of Rossby waves in two-dimensional flow on a β-plane. *Physica D*, 245: 1–7, 2013.